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ON THE MOTION OF ISO-SURFACES IN A FLUID MEDIUM*

By

S. K. Banerji, *Calcutta.*

A method of analysis which has been found to be of great value in forecasting changes of conditions in the atmosphere may be of interest to students of fluid dynamics. The energy transformation that takes place in the atmosphere due to evaporation, condensation, radiation, absorption of solar heat introduces very great complexity in the dynamics of air motion. Yet it is possible to calculate the motion of iso-surfaces for short periods without recourse to complicated mathematics. By an iso-surface I mean an isobaric surface or isothermal surface or any surface along which the value of a meteorological element remains constant

If the axis of x be drawn towards the east, the axis of y towards the north and the axis of z vertically upwards, then

$$p = p(x, y, z, t) = \text{constant} = C,$$

will represent an isobaric surface at any instant t. We assume that the surface is moving in a continuous manner and we want to find the displaced position of the surface after a given interval of time. We need not consider the displacement of an element of the surface tangential to it, because the tangential displacement is of no consequence so far as the future position of the isobaric surface is concerned. We need only consider the motion of the surface normal to itself.

If V denotes the velocity of a point on the surface normal to itself, we get at time $t+\delta t$,

$$p(x+lV\delta t,\ y+mV\delta t,\ z+nV\delta t,\ t+\delta t) = C,$$

where l, m, n are the direction cosines of the normal at (x, y, z) on the surface.

Therefore,

$$V\left(l\frac{\partial p}{\partial x}+m\frac{\partial p}{\partial y}+n\frac{\partial p}{\partial z}\right)+\frac{\partial p}{\partial t} = 0,$$

or

$$V = -\frac{\partial p}{\partial t}\Big/\frac{\partial p}{\partial n},$$

where δn is an element of the normal to the surface at (x, y, z). The velocity, therefore, varies directly as the barometric tendency and inversely as the pressure gradient.

If $A_1, A_2, A_3, \ldots$ represent the first, second, third, ... differential coefficients respectively of the displacement S of the isobaric surface normal to itself, then A_1 is equal to the velocity V, A_2 is the acceleration, etc. Now,

---

* An address delivered at the Annual Meeting of the Calcutta Mathematical Society in January, 1956.

$$A_2 = \frac{\partial V}{\partial t}$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + V\frac{\partial}{\partial n}$$

$$\frac{D^2}{Dt^2} = \left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial n}\right)\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial n}\right)$$

$$= \frac{\partial^2}{\partial t^2} + 2V\frac{\partial^2}{\partial n\partial t} + V^2\frac{\partial^2}{\partial n^2} + \frac{\partial V}{\partial t}\frac{\partial}{\partial n}$$

$$= \frac{\partial^2}{\partial t^2} + 2V\frac{\partial^2}{\partial n\partial t} + V^2\frac{\partial^2}{\partial n^2} + A_2\frac{\partial}{\partial n}.$$

We see at once that with a system of axes fixed to the moving surface in question,

$$\frac{Dp}{Dt} = 0, \quad \frac{D^2p}{Dt^2} = 0.$$

Therefore $$\frac{\partial p}{\partial t} + V\frac{\partial p}{\partial n} = 0$$

and $$\frac{\partial^2 p}{\partial t^2} + 2V\frac{\partial^2 p}{\partial n\partial t} + V^2\frac{\partial^2 p}{\partial n^2} + A_2\frac{\partial p}{\partial n} = 0.$$

Hence $$A_2 = -\left(\frac{\partial^2 p}{\partial t^2} + 2V\frac{\partial^2 p}{\partial n\partial t} + V^2\frac{\partial^2 p}{\partial n^2}\right)\Big/\frac{\partial p}{\partial n}.$$

In a similar way we can find the third, A_3, the fourth, A_4, and higher differential coefficients with respect to t.

The displacement S of the isobaric surface along the normal will be given by

$$S = Vt + \tfrac{1}{2}A_2t^2 + \tfrac{1}{6}A_3t^3 + \ldots$$

If we want the displacement after a short interval, the first two terms will give fairly close approximation.

If observations of pressure be taken every three hours or every six hours of G. M. T., we can draw charts giving the isobaric surfaces every three hours or every six hours. When we have three such consecutive surfaces, it is possible to obtain the values of $\frac{\partial p}{\partial t}$, $\frac{\partial p}{\partial n}$, $\frac{\partial^2 p}{\partial t^2}$, $\frac{\partial^2 p}{\partial n\partial t}$, $\frac{\partial^2 p}{\partial n^2}$ and to calculate V and A_2 with the help of the formulae given above. Consequently we can determine the form of the isobaric surface at any time following these three observations but before the time of the next observation.

Similar procedure can be adopted for isothermal surfaces or any other iso-surfaces.

If we want to determine the motion of a point where the pressure becomes maximum or minimum (or what in meteorological language is called an anticyclone centre or a cyclone centre) the formulae would need some modification.

In this case, if the axes be drawn with reference to the singular point,

$$\frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = 0.$$

The second order differential coefficients satisfy the conditions

$$\frac{\partial^2 p}{\partial x^2} \gtrless 0, \quad \frac{\partial^2 p}{\partial y^2} \gtrless 0, \quad \frac{\partial^2 p}{\partial z^2} \gtrless 0,$$

according as the pressure at the point is minimum or maximum.

If V_x, V_y, V_z denote the components of velocity of the singular point, we get

$$V_x = -\frac{\partial^2 p}{\partial x \partial t} \Big/ \frac{\partial^2 p}{\partial x^2}, \quad V_y = -\frac{\partial^2 p}{\partial y \partial t} \Big/ \frac{\partial^2 p}{\partial y^2}, \quad V_z = -\frac{\partial^2 p}{\partial z \partial t} \Big/ \frac{\partial^2 p}{\partial z^2}.$$

The velocity of the pressure centre is, therefore, proportional to the isalobaric gradient and inversely proportional to the steepness of the pressure profile.

The components of acceleration A_x, A_y, A_z are easily found to be

$$A_x = -\left(\frac{\partial^3 p}{\partial x \partial t^2} \cdot \frac{\partial^2 p}{\partial x^2} - 2 \frac{\partial^2 p}{\partial x \partial t} \cdot \frac{\partial^3 p}{\partial x^2 \partial t} \right) \Big/ \left(\frac{\partial^2 p}{\partial x^2} \right)^2$$

$$A_y = -\left(\frac{\partial^3 p}{\partial y \partial t^2} \cdot \frac{\partial^2 p}{\partial y^2} - 2 \frac{\partial^2 p}{\partial y \partial t} \cdot \frac{\partial^3 p}{\partial y^2 \partial t} \right) \Big/ \left(\frac{\partial^2 p}{\partial y^2} \right)^2$$

$$A_z = -\left(\frac{\partial^3 p}{\partial z \partial t^2} \cdot \frac{\partial^2 p}{\partial z^2} - 2 \frac{\partial^2 p}{\partial z \partial t} \cdot \frac{\partial^3 p}{\partial z^2 \partial t} \right) \Big/ \left(\frac{\partial^2 p}{\partial z^2} \right)^2.$$

In the atmosphere, the pressure decreases with height. It is, therefore, usual to determine the motion of the pressure centre on a horizontal surface.

Movement of surfaces of discontinuity. The surface of discontinuity, which is supposed to be a surface of separation between two air masses of different densities, is of finite thickness and is inclined at an angle θ to the horizon depending on the difference of the densities. If p_1, p_2 denote the pressures at the two sides of the surface of discontinuity

$$\frac{\partial p_1}{\partial x} - \frac{\partial p_2}{\partial x} = -g(\varrho_1 - \varrho_2),$$

ϱ_1, ϱ_2 being the densities of the air masses. So long as the front (or discontinuity) maintains its characteristic existance, ϱ_1 is never equal to ϱ_2. Therefore, $\partial p_1/\partial x$ is never equal to $\partial p_2/\partial x$.

At the surface of discontinuity, the following conditions must be satisfied:—

(1) The pressure on the two sides of the surface must be equal, but not the pressure gradients.

(2) The velocity components (V_1, V_2) normal to the surface must be equal on both sides.

Thus $$p_1 - p_2 = 0, \quad V_1 - V_2 = 0.$$

The velocity of the front in the direction of the normal is given by

$$V_1 = -\left(\frac{\partial p_1}{\partial t} - \frac{\partial p_2}{\partial t}\right) \Big/ \left(\frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n}\right).$$

The velocity is greater the greater is the difference between the tendencies on the two sides of the front, and the smaller is the difference between the pressure gradients. A front is stationary only when

$$\frac{\partial p_1}{\partial t} = \frac{\partial p_2}{\partial t},$$

or the tendencies are equal on both sides of the front. The acceleration A of the front is given by

$$A = -\left[\left(\frac{\partial^2 p_1}{\partial t^2} - \frac{\partial^2 p_2}{\partial t^2}\right) + 2V\left(\frac{\partial^2 p_1}{\partial n \partial t} - \frac{\partial^2 p_2}{\partial n \partial t}\right) + V^2\left(\frac{\partial^2 p_1}{\partial n^2} - \frac{\partial^2 p_2}{\partial n^2}\right)\right] \Big/ \left(\frac{\partial p_1}{\partial n} - \frac{\partial p_2}{\partial n}\right).$$

Deepening and filling up of a depression. Deepening or filling refers to the pressure in a moving system of coordinates which is fixed to the moving pressure system on a horizontal surface. The connection between the pressure variation in a moving system of coordinates and a fixed system is given by

$$\frac{Dp}{Dt} = \frac{\partial p}{\partial t} + V \cdot \nabla p.$$

This equation shows that the pressure variation at a fixed station is composed of two parts: (1) one which is caused by the movement of the pressure system, and (2) one which is caused by deepening or filling. In a moving system of coordinates the pressure variation when expanded in Maclaurin's series is given by

$$\Delta p = \left(\frac{Dp}{Dt}\right)_0 t + \tfrac{1}{2}\left(\frac{D^2 p}{Dt^2}\right)_0 t^2,$$

neglecting terms of the third and higher orders.

The expression is equivalent to

$$\Delta p = \left(\frac{\partial p}{\partial t} + V \cdot \nabla p\right)_0 t + \tfrac{1}{2}\left[\frac{\partial^2 p}{\partial t^2} + 2V \cdot \nabla \frac{\partial p}{\partial t} + \frac{\partial V}{\partial t} \cdot \nabla p + V \cdot \nabla (V \cdot \nabla p)\right]_0 t^2.$$

At a pressure centre, $$[\nabla p]_0 = 0.$$

For a horizontal surface, Δp reduces to the form

$$\Delta p = \left(\frac{\partial p}{\partial t}\right)_0 t + \tfrac{1}{2}\left[\frac{\partial^2 p}{\partial t^2} + 2V_x \frac{\partial^2 p}{\partial x \partial t} + 2V_y \frac{\partial^2 p}{\partial y \partial t} + V_x^2 \frac{\partial^2 p}{\partial x^2} + V_y^2 \frac{\partial^2 p}{\partial y^2}\right]_0 t^2 + V_x V_y \left(\frac{\partial^2 p}{\partial x \partial y}\right)_0 t^2$$

When the axes are chosen to be two lines along which there is either maximum or minimum of curvature of the pressure profiles, the last term vanishes. The integral of deepening or filling per unit time over a surface is given by

$$\int \frac{Dp}{Dt}\,d\sigma = \int \frac{\partial p}{\partial t}\,d\sigma + \int V_c \cdot \nabla p\,d\sigma$$

If i, j be unit vectors and i coincides with ∇p, we have

$$\nabla p = (\partial p/\partial h)\,i.$$

If $dp = 1$, then dh is the distance between isobars differing by one unit of pressure, say, a millibar. Also

$$d\sigma = dh\,ds$$

ds being an element of length of isobar.

Therefore,
$$\int V_c \cdot \nabla p\,d\sigma = \int V_c \cdot i\,ds = V_o \cdot i \int ds = 0.$$

Hence
$$\int \frac{Dp}{Dt}\,d\sigma = \int \frac{\partial p}{\partial t}\,d\sigma.$$

The deepening or filling over the area between two closed isobars is thus equal to the planimetric value of the barometric tendency in the same area.

In a system of coordinates fixed to a moving isobar

$$Dp/Dt = 0.$$

Therefore,
$$\partial p/\partial t = -V \cdot \nabla p.$$

Hence if isobars be drawn at intervals of unit differences of pressure, so that $dp = 1$,

$$\int \frac{\partial p}{\partial t}\,d\sigma = -\int V \cdot \nabla p\,d\sigma = \int \left(-V \cdot \frac{\partial p}{\partial h}\,i\right) dh\,ds = -\int V \cdot i\,ds = -\frac{dA}{dt},$$

where A is the area enclosed by the isobar.

Therefore, when the area enclosed by a closed isobar is contracting, the depression is deepening and when the area is expanding, it is filling up.

These and other derivations that can be made by pursuing this method of analysis have wide applications in atmospheric motions.

Motion of deformable systems. These principles can also be applied in finding approximate solutions of problems involving the motion of deformable surfaces in fluids.

Consider, for instance, the problem of the horizontal motion of a spherical mass of air initially enclosed in a spherical soap bubble and moving through still air with an initial velocity U_0. Although apparently simple, its solution involves complicated mathematics. We can proceed to get approximate solutions from the following considerations.

The fact that initially the bubble had spherical form means that it had uniform pressure all over its surface. At $t = 0$, the isobaric surface is therefore a sphere of radius a, a being the radius of the spherical bubble.

Since according to the general principle, the initial motion may be considered to be that due to a double source, the velocity potential φ has initially the form

$$\varphi = A\frac{\partial}{\partial x}\frac{e^{-\iota kr}}{r} \cdot e^{\iota\sigma t} = Ae^{\iota\sigma t}\frac{d}{dr}\frac{e^{-\iota kr}}{r}\cos\theta,$$

where $\sigma = k/c$, c being the velocity of sound. If T be the time taken by the moving mass of air to come to rest, T will be equal to the quarter period, and therefore $T = \frac{1}{2}\pi/\sigma$.

We have also $-\frac{\partial\varphi}{\partial r} = U\cos\theta$ at $r = a$ and $U = U_0 e^{\iota\sigma t}$.

Hence

$$A\frac{d^2}{da^2}\cdot\frac{e^{\iota ka}}{a} = -U_0,$$

or

$$A = \frac{(2 - k^2a^2 - 2\iota ka)U_0a^3e^{\iota ka}}{4 + k^4a^4}.$$

The pressure p on the surface is given by

$$\begin{aligned} p &= p_0 + \varrho_0\dot{\varphi} \\ &= p_0 + \iota\sigma\varrho_0 A.\frac{d}{da}\frac{e^{-\iota ka}}{a}.\cos\theta\, e^{\iota\sigma t} \\ &= p_0 + \frac{\varrho_0}{4 + k^4a^4}\Big[(4ka^2 - k^3a^4)\sigma U \\ &\qquad + (k^2a^3 - 2k^3a^4 - 2a)\frac{dU}{dt}\Big]\cos\theta. \end{aligned}$$

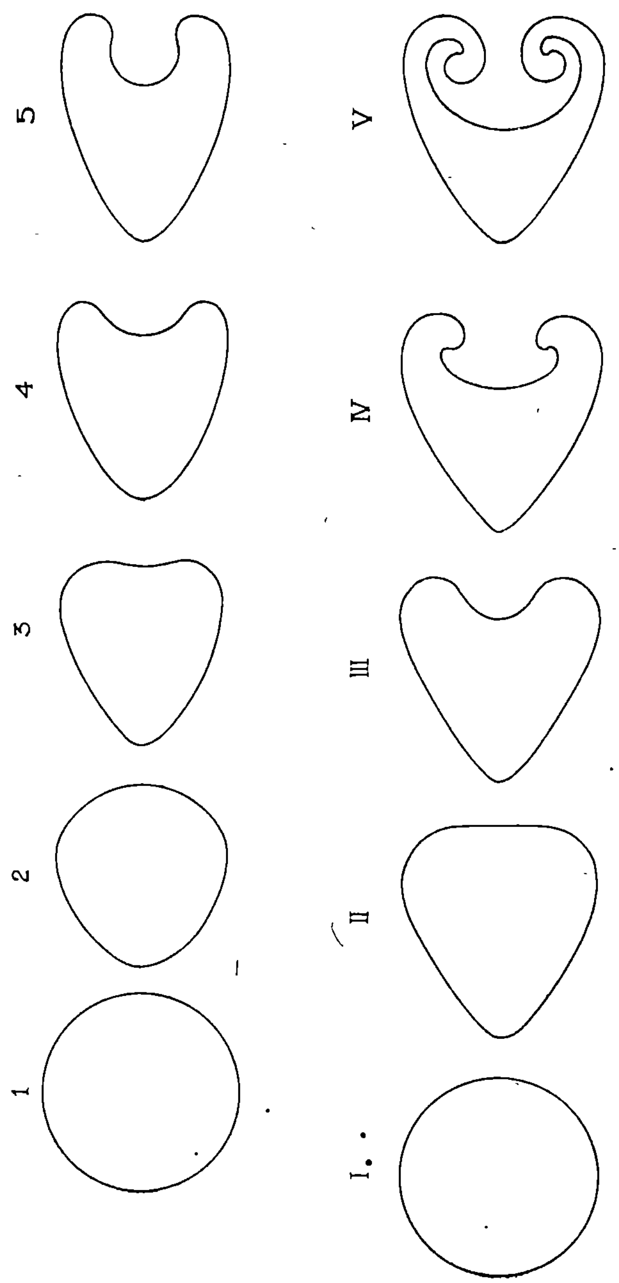
The wave-length $2\pi/k$ is large compared to a. Therefore all the terms in the above expression involving k can be neglected. Hence,

$$p = p_0 - \tfrac{1}{2}\varrho_0 a\frac{dU}{dt}\cos\theta.$$

The expression shows clearly that the pressure gradient, $p_0 - p$, is positive in the direction $\theta = 0$ and negative in the opposite direction $\theta = \pi$.

At a point (r, θ), the pressure is given by

$$p = p_0 + \iota\sigma\varrho_0 A\frac{d}{dr}\frac{e^{-\iota kr}}{r}\cos\theta . e^{\iota\sigma t}.$$



1, 2, 3, 4, 5: COMPUTED FORMS.

I, II, III, IV, V: OBSERVED FORMS.

The expression reduces to the form

$$p = p_0 + \tfrac{1}{2}\varrho_0 a^3 \left[\frac{\sin k(r-a)}{r}\, \sigma U - \frac{\cos k(r-a)}{r^2}\, \frac{dU}{dt} \right] \cos\theta .$$

For points near the spherical surface $k(r-a)$ is a small quantity.

Therefore at such points,

$$p = p_0 - \tfrac{1}{2}\varrho_0 a^3 \frac{1}{r^2}\, \frac{dU}{dt} \cos\theta .$$

This expression enables us to draw the isobaric surface $p = \text{constant} = p_1$, after an interval of 1 sec., that is, at $t = 1$ sec. In making the computation, U is taken as $U = U_0 \cos\sigma t$, involving the real part of the time-factor, and a, U_0, T to have the experimentally observed values. From this isobaric surface and the isobaric surface at $t = 0$, we can calculate the velocities at a number of points on the isobaric surface. Since $\delta t = 1$ sec., these velocities are proportional to the normal distance between the two isobaric surfaces. Using these velocities we can get the displaced position and the form of the isobaric surface at $t = 2$ sec. Proceeding in this way we can get the positions of the isobaric surfaces after each interval of a second. The method, however, involves greater and greater errors in approximation in each successive step. The observed shapes of a section of a spherical mass of coloured air ejected horizontally through still air, and the shapes of sections of computed isobaric surfaces are shown in the figure (see page 7). We observe that the movement disturbs the initial equality of pressure on the surface. The continued deformation that takes place is the result of continued attempt to reach equality of pressure again on the surface. When this is attained finally, no further deformation takes place.

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TRANSFORMATIONS OF HYPERGEOMETRIC FUNCTIONS OF THREE VARIABLES

By

Santi Saran, *Lucknow, U. P.*

(*Communicated by Dr. S. C. Mitra—Received April 29, 1954*)

1. There are several methods for obtaining transformations of hypergeometric functions of three variables.

Firstly, by transformation of the hypergeometric series. When the triple series is rewritten as an infinite sum of hypergeometric functions of two variables, the known transformation theory of such functions can be applied to each term. This method is quite simple and, in a limited range, very effective for discovering transformations as well as proving them.

Secondly, by transformation of the systems of partial differential equations satisfied by the hypergeometric functions. This method, though simple in theory, is rather laborious in practice and not very useful for discovering new transformations.

Thirdly, by transformation of the integrals representing the hypergeometric functions. This third method will be applied here to prove a number of transformations. Some of these transformations have previously been obtained by the author [Santi Saran,] by the first method.

The functions F_G, F_M, F_N, F_S and F_T are defined as.

$$F_G(\alpha'_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma'_2; x, y, z)$$

$$=\sum\frac{(\alpha_1, m+n+p)(\beta_1, m)(\beta_2, n)(\beta_3, p)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)}x^m y^n z^p \tag{1}$$

$$F_M(\alpha_1, \alpha'_2, \beta'_1, \beta_2; \gamma_1, \gamma'_2; x, y, z)$$

$$=\sum\frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)}x^m y^n z^p \tag{2}$$

$$F_N(\alpha_1, \alpha_2, \alpha_3, \beta'_1, \beta_2; \gamma_1, \gamma'_2; x, y, z)$$

$$=\sum\frac{(\alpha_1, m)(\alpha_2, n)(\alpha_3, p)(\beta_1, m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m)(\gamma_2, n+p)}x^m y^n z^p \tag{3}$$

$$F_S(\alpha_1, \alpha'_2, \beta_1, \beta_2, \beta_3; \gamma'_1; x, y, z)$$

$$=\sum\frac{(\alpha_1, m)(\alpha_2\ n+p)(\beta_1, m)(\beta_2, n)(\beta_3\ p)}{(1, m)(1, n)(1, p)(\gamma_1, m+n+p)}x^m y^n z^p \tag{4}$$

and

$$F_T(\alpha_1, \alpha'_2, \beta'_1, \beta_2, \gamma'_1; x, y, z)$$

$$=\sum\frac{(\alpha_1, m)(\alpha_2, n+p)(\beta_1. m+p)(\beta_2, n)}{(1, m)(1, n)(1, p)(\gamma_1, m+n+p)}x^m y^n z^p \tag{5}$$

where $(A, S) \equiv \dfrac{\Gamma(A+S)}{\Gamma(A)}$, $\sum$ takes the values of m, n, p from zero to infinity γ's should not be negative integers. The numbers (1, 2, 3) are associated with (m, n, p) so that $F(\alpha_1\ \alpha_3)$ means $(\alpha_1, m)\ (\alpha_3, p)$. Dash (′) is used to denote more than one sum, for example, $F(\alpha'_1, \alpha_3)$ mean $(\alpha_1, m+n)(\alpha_3, p)[\neq(\alpha_1, m+p)(\alpha_3, n)$ because α_3 is associated to $p]$. Similarly $F(\alpha_1, \alpha'_2)$ means $(\alpha_1, m)(\alpha_2, n+p)$. Also $F(\alpha'_1)$ gives $(\alpha_1, m+n+p)$. Dashes only indicate the places and therefore they are not marked on the right hand side. Further if $|x|<r$, $|y|<s$ and $|z|<t$, the domain of convergence of each function is given by

$$F_G \ldots\ldots\ldots \left.\begin{array}{r} r+s=1 \\ r+t=1 \end{array}\right\} \qquad F_M \ldots\ldots\ldots \left.\begin{array}{r} r+t=1 \\ s=1 \end{array}\right\}$$

$$F_N \ldots\ldots s(1-r)+t(1-s)=0, \quad F_S \ldots\ldots \frac{1}{r}+\frac{1}{s}=\frac{1}{r}+\frac{1}{t}=1$$

and $$F_T \ldots\ldots t = r - rs + s.$$

The integral representations [Santi Saran, *l. c.*] are given by

$$\frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)\Gamma(\gamma_2-\beta_2-\beta_3)\Gamma(\gamma_1-\beta_1)}{\Gamma(\gamma_1)\Gamma\gamma_2)} F_G$$

$$\iiint u^{\beta_1-1} v^{\beta_2-1} w^{\beta_3-1} (1-u)^{\gamma_1-\beta_1-1} (1-v-w)^{\gamma_2-\beta_2-\beta_3-1} (1-ux-vy-wz)^{-\alpha_1}\, du\,dv\,dw$$

$$R(\gamma_1) > R(\beta_1) > 0,\ R(\gamma_2) > R(\beta_2+\beta_3) > 0 \text{ where } R(\beta_2) > 0,$$
$$R(\beta_3) > 0,\ v \geq 0,\ w \geq 0,\ v+w \leq 1,\ \varrho+\varrho'+\varrho'' < 1.$$

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\gamma_1-\alpha_1)\Gamma(\gamma_2-\alpha_2)}{\Gamma(\gamma_1)\Gamma(\gamma_2)} F_M$$

$$= \int_0^1 \int_0^1 u^{\alpha_1-1} v^{\alpha_2-1} (1-u)^{\gamma_1-\alpha_1-1} (1-v^{\gamma_2-\alpha_2-1} (1-vy)^{-\beta_2} (1-ux-vz)^{-\beta_1} du\,dv$$
$$R(\gamma_1) > R(\alpha_1) > 0,\ R(\gamma_2) > R(\alpha_2) > 0,\ \varrho+\varrho'' < 1$$

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)\Gamma(\gamma_1-\alpha_1)\Gamma(\gamma_2-\alpha_2-\alpha_3)}{\Gamma(\gamma_1)\Gamma(\gamma_2)} F_N$$

$$= \iiint u^{\alpha_1-1} v^{\alpha_2-1} w^{\alpha_3-1} (1-u)^{\gamma_1-\alpha_1-1} (1-v-w)^{\gamma_2-\alpha_2-\alpha_3-1} (1-vy)^{-\beta_2} (1-ux-wz)^{-\beta_1} du\,dv\,dw$$
$$R(\gamma_1) > R(\alpha_1) > 0,\ R(\gamma_2) > R(\alpha_2+\alpha_3) > 0 \text{ where } R(\alpha_2) > 0,\ R(\alpha_3) > 0,$$
$$\varrho+\varrho'' < 1,\ u \geq 0,\ w \geq 0,\ v+w \leq 1.$$

$$\frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)\Gamma(\gamma_1-\beta_1-\beta_2-\beta_3)}{\Gamma(\gamma_1)} F_S$$

$$= \iiint u^{\beta_1-1} v^{\beta_2-1} w^{\beta_3-1} (1-u-v-w)^{\gamma_1-\beta_1-\beta_2-\beta_3-1} (1-ux)^{-\alpha_1} (1-vy-wz)^{-\alpha_2} du\,dv\,dw$$
$$R(\gamma_1) > R(\beta_1+\beta_2+\beta_3) > 0 \text{ where } R(\beta_1) > 0,\ R(\beta_2) > 0,\ R(\beta_3) > 0$$
$$u \geq 0,\ v \geq 0,\ w \geq 0,\ u+v+w \leq 1,\ \varrho'+\varrho'' < 1.$$

Also $\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\gamma_1-\alpha_1-\alpha_2)}{\Gamma(\gamma_1)} F_S$

$$=\iint u^{\alpha_1-1}v^{\alpha_2-1}(1-u-v)^{\gamma_1-\alpha_1-\alpha_2-1}(1-ux)^{-\beta_1}(1-vy)^{-\beta_2}(1-vz)^{-\beta_3}dudv$$

$R(\gamma_1)>R(\alpha_1+\alpha_2)>0$ where $R(\alpha_1)>0$, $R(\alpha_2)>0$

$u\geq 0,\ v\geq 0,\ u+v\leq 1.$

and $\frac{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\gamma_1-\beta_1-\beta_2)}{\Gamma(\gamma_1)} F_T$

$$=\iint u^{\beta_1-1}v^{\beta_2-1}(1-u-v)^{\gamma_1-\beta_1-\beta_2-1}(1-vx)^{-\alpha_1}(1-vy-vz)^{-\alpha_2}dudv$$

$R(\gamma_1)>R(\beta_1+\beta_2)>0$ where $R(\beta_1)>0$, $R(\beta_2)>0$

$u\geq 0,\ v\geq 0,\ u+v\leq 1.$

Throughout the integrals it has been assumed that $|x|\leq\varrho$, $|y|\leq\varrho'$ and $|z|\leq\varrho''$

2. Consider the integral

$$\iiint u^{\beta_1-1}v^{\beta_2-1}w^{\beta_3-1}(1-u)^{\gamma_1-\beta_1-1}(1-v-w)^{\gamma_2-\beta_2-\beta_3-1}(1-ux-vy-wz)^{-\alpha_1}dudvdw$$

Put $v=s(1-t)$, $w=st$, the integral becomes.

$$\int_0^1\int_0^1\int_0^1 u^{\beta_1-1}s^{\beta_2+\beta_3-1}t^{\beta_3-1}(1-u)^{\gamma_1-\beta_1-1}(1-s)^{\gamma_2-\beta_2-\beta_3-1}(1-t)^{\beta_2-1}\times$$
$$(1-ux-sy+sty-stz)^{-\alpha_1}dudsdt$$

Now $(1-ux-sy+sty-stz)^{-\alpha_1}$

(i) $$=\sum_{m=0}^{\infty}\frac{(\alpha_1,\ m)}{(1,\ m)}(z-y)^m s^m t^m(1-ux-sy)^{-\alpha_1-m} \text{ if } \left|\frac{z-y}{1-x-y}\right|<1$$

(ii) $$=\sum_{m=0}^{\infty}\frac{(\alpha_1,\ m)}{(1,\ m)}s^m y^m(1-ux)^{-\alpha_1-m}\left\{1-t\left(1-\frac{z}{y}\right)\right\}^m \text{ if } \left|\frac{y}{1-x}\right|+\left|\frac{z}{1-x}\right|<1$$

using (i) if $\left|\frac{z-y}{1-x-y}\right|<1$, we obtain

$$\sum_{m=0}^{\infty}\frac{(\alpha_1,\ m)}{(1,\ m)}(z-y)^m\int_0^1\int_0^1\int_0^1 u^{\beta_1-1}s^{\beta_2+\beta_3+m-1}t^{\beta_3+m-1}(1-u)^{\gamma_1-\beta_1-1}(1-s)^{\gamma_2-\beta_2-\beta_3-1}\times$$
$$(1-t)^{\beta_2-1}(1-ux-sy)^{-\alpha_1-m}dudsdt,$$

Evaluating the integral for t and using the integral for F_2 [Bailey, *l.c,*] namely

$$\frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}{\Gamma(\gamma)\Gamma(\gamma')} F_2(\alpha,\ \beta,\ \beta';\ \gamma,\ \gamma';\ x,\ y)$$

$$= \int_0^1 \int_0^1 u^{\beta-1} v^{\beta'-1} (1-u)^{\gamma-\beta-1} (1-v)^{\gamma'-\beta'-1} (1-ux-vy)^{-\alpha} du dv$$

provided $R(\gamma) > R(\beta) > 0$. $R(\gamma^1) > R(\beta^1) > 0$.

we obtain, if $R(\gamma_1) > R(\beta_1) > 0$, $R(\gamma_2) > R(\beta_2+\beta_3) > 0$ where

$R(\beta_2) > 0$, $R(\beta_3) > 0$ and $\left|\frac{z-y}{1-x-y}\right| < 1$

$$\sum_{m=0}^{\infty} \frac{(\alpha_1, m)}{(1, m)} (z-y)^m \frac{\Gamma(\beta_3+m)\Gamma(\beta_2)}{\Gamma(\beta_2+\beta_3+m)} \cdot \frac{\Gamma(\beta_1)\Gamma(\beta_2+\beta_3+m)\Gamma(\gamma_1-\beta_1)\Gamma(\gamma_2-\beta_2-\beta_3)}{\Gamma(\gamma_1)\Gamma(\gamma_2+m)}$$

$$\times F_2(\alpha_1+m, \beta_1, \beta_3+\beta_3+m; \gamma_1, \gamma_3+m; x, y).$$

Hence $F_G(\alpha'_1, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma'_2; x, y, z)$.

$$\sum_{m=0}^{\infty} \frac{(\alpha_1, m)(\beta_3, m)}{(1, m)(\gamma_2, m)} (z-y)^m F_2(\alpha_1+m, \beta_1, \beta_2+\beta_3+m; \gamma_1, \gamma_2+m; x, y)$$

Expanding the series on the right hand side and putting $y = z$, we get

$$F_G(\alpha'_1, \beta_1. \beta_2, \beta_3; \gamma_1, \gamma'_2; x, y, y)$$
$$= F_2(\alpha_1; \beta_1, \beta_2+\beta_3; \gamma_1, \gamma_2; x, y). \quad \text{[Santi Saran, } l.c.\text{]} \qquad (2,1a)$$

Further using (ii) we obtain, if $\left|\frac{y}{1-x}\right| + \left|\frac{z}{1-x}\right| < 1$.

$$\int_0^1 \int_0^1 \int_0^1 u^{\beta_1-1} s^{\beta_2+\beta_3-1} t^{\beta_3-1} (1-u)^{\gamma_1-\beta_1-1} (1-s)^{\gamma_2-\beta_2-\beta_3-1} (1-t)^{\beta_2-1} \times$$

$$\sum_{m=0}^{\infty} \frac{(\alpha_1, m)}{(1, m)} s^m y^m (1-ux)^{-\alpha_1-m} \left\{1-\left(1-\frac{z}{y}\right)\right\}^m du ds dt$$

Evaluating the integral for s and using the relation [Appell et Kampé de Ferriet, 1926]

$$\frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} {}_2F_1(\alpha, \beta; \gamma; x)$$

$$= \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta} du, \ R(\gamma) > R(\alpha) > 0$$

we obtain $F_G(\alpha'_1, \beta_1, \beta_3; \gamma_1, \gamma'_2; x, y, z)$

$$= \sum_{m=0}^{\infty} \frac{(\alpha_1, m)(\beta_2+\beta_3, m)}{(1, m)(\gamma_2, m)} {}_2F_1(\alpha_1+m, \beta_1; \gamma_1; x) {}_2F_1(-m, \beta_3; \beta_2+\beta_3; 1-z/y) y^m \qquad (2.2)$$

provided $\left|\frac{y}{1-x}\right| + \left|\frac{z}{1-x}\right| < 1$,

being the condition for changing the order of integration and summation using Euler's transfomation [Appell *et al.*, 1926]

$$ {}_2F_1(a,\ b\ ;\ c\ ;\ z) = (1-z)^{-a}\,{}_2F_1(a,\ c-b\ ;\ c\ ;\ -z/1-z) $$

we obtain, after applying to each of the hypergeometric function ${}_2F_1$.

$$ = (1-x)^{-\alpha_1} y^{\beta_3} z^{-\beta_3} F_F\left(\gamma_1-\beta_1,\ \beta_2+\beta'_3,\ \alpha_1\ \beta_3'^{*}\ ;\ \gamma_1,\ \gamma_2,\ \beta_2+\beta_3\ ;\ \frac{-x}{1-x},\ \frac{y}{1-x},\ \left(1-\frac{y}{x}\right)\right) \tag{2.3} $$

$$ = z^{\beta_3} y^{-\beta_3} \sum_{m=0}^{\infty} \frac{(\alpha_1,\ m)(\beta_2+\beta_3,\ m)}{(1,\ m)(\gamma_2,\ m)}\ {}_2F_1(\alpha_1+m,\ \beta_1\ ;\ \gamma_1\ ;\ x) \times $$

$$ \times\,{}_2F_1(\beta_2+\beta_3+m,\ \beta_3\ ;\ \beta_2+\beta_3\ ;\ 1-y/z)y^{m} \tag{2.4} $$

Similar other transformations follow by using the other Euler's transformations [Appell *et al.*, *l.c.*]

In (2.2) and (2.4), putting $x = 0$ or $y = z$ we obtain the transformations (1), (9) [Appell *et al.*, *l.c.*, p. 35] but in (2.3) we only get (9) [Appell *et ai.*, *l.c.*,]

Further if we make substitution

$$ v = 1-s,\ \ w = st $$

the integral takes the form

$$ \int_0^1\int_0^1\int_0^1 u^{\beta_1-1} s^{\gamma_2-\beta_2-1} t^{\beta_3-1}(1-u)^{\gamma_1-\beta_1-1}(1-s)^{\beta_2-1}(1-t)^{\gamma_2-\beta_2-\beta_3-1}\times $$

$$ (1-y-ux+sy-stz)^{-\alpha_1}\,du\,ds\,dt $$

Here $(1-y-ux+sy-stz)^{-\alpha_1}$

$$ = \sum_{m=0}^{\infty}\frac{(\alpha_1,\ m)}{(1,\ m)} s^{m}(-y)^{m}(1-y-ux)^{-\alpha_1-m}\left(1-\frac{z}{y}t\right)^{m} $$

provided

$$ \left|\frac{y}{1-x-y}\right| + \left|\frac{z}{1-x-y}\right| < 1 $$

under this condition, after evaluating the integral for s we obtain

$$ F_G(\alpha'_1\ ;\ \beta_1,\ \beta_2,\ \beta_3\ ;\ \gamma_1,\ \gamma_2'\ ;\ x,\ y,\ z), $$

$$ = (1-y)^{-\alpha_1}\sum_{m=0}^{\infty}\frac{(\alpha_1,\ m)(\gamma_2-\beta_2,\ m)}{(1,\ m)(\gamma_2,\ m)}\cdot{}_2F_1\left((\alpha_1+m,\ \beta_1\ ;\ \gamma_1\ ;\ \frac{x}{1-y}\right)\times $$

$$ {}_2F_1\left(-m,\ \beta_3\ ;\ \gamma_2-\beta_2;\ \frac{z}{y}\right)\left(\frac{y}{y-1}\right)^{m} \tag{2.5} $$

If $z = y$, then by virtue of (2.1a) we get result (6) [Bailey, 1935, p. 78]

Again using Euler's transformations [Appell *et al.*, *l.c.*] a number of transformations follow.

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* $(\beta_3,\ m+p)$

3. Again consider the integral

$$\iiint u^{\alpha_1-1}v^{\alpha_2-1}w^{\alpha_3-1}(1-u)^{\gamma_1-\alpha_1-1}(1-v-w)^{\gamma_2-\alpha_2-\alpha_3-1}(1-vy)^{-\beta_2}(1-ux-wz)^{-\beta_1}du\,dv\,dw$$

$R(\gamma_1) > R(\alpha_1) > 0$, $R(\gamma_2) > R(\alpha_2+\alpha_3) > 0$ where $R(\gamma_2) > 0$, $R(\alpha_3) > 0$

$v \geq 0$, $w \geq 0$, $v+w \leq 1$.

Making the substitution $v = s(1-t)$, $w = st$, the integral becomes

$$\int_0^1\int_0^1\int_0^1 u^{\alpha_1-1}s^{\alpha_2+\alpha_3-1}t^{\alpha_3-1}(1-u)^{\gamma_1-\alpha_1-1}(1-s)^{\gamma_2-\alpha_2-\alpha_3-1}(1-t)^{\alpha_2-1}\times$$

$$(1-sy+sty)^{-\beta_2}(1-ux-stz)^{-\beta_1}du\,ds\,dt$$

now

$$(1-sy+sty)^{-\beta_2} = [1-sy(1-t)]^{-\beta_2}$$

$$=\sum_{m=0}^{\infty}\frac{(\beta_2, m)}{(1, m)}s^m y^m(1-t)^m$$

valid if $|y| < 1$, the change in the order of integration and summation will, therefore, be valid under this condition and therefore we obtain

$$\sum_{m=0}^{\infty}\frac{(\beta_1, m)}{(1, m)}y^m\int_0^1\int_0^1\int_0^1 u^{\alpha_1-1}s^{\alpha_2+\alpha_3+m-1}t^{\alpha_3-1}(1-u)^{\gamma_1-\alpha_1-1}(1-s)^{\gamma_2-\alpha_2-\alpha_3-1}(1-t)^{\alpha_2+m-1}\times$$

$$(1-ux-stz)^{-\beta_1}du\,ds\,dt$$

using the integral for F_2 [Bailey, *l.c.*] namely

$$\frac{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}{\Gamma(\gamma)\,\Gamma(\gamma')}F_2(\alpha\,;\,\beta,\,\beta'\,;\,\gamma,\,\gamma'\,;\,x,\,y)$$

$$=\int_0^1\int_0^1 u^{\beta-1}v^{\beta'-1}(1-u)^{\gamma-\beta-1}(1-v)^{\gamma'-\beta'-1}(1-ux-vy)^{-\alpha}du\,dv\,\ldots\ldots\ldots\ldots \quad (1)$$

valid if $R(\gamma) > R(\beta) > 0$, $R(\gamma') > R(\beta') > 0$.

Then $\quad \dfrac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\alpha_2+\alpha_3)}F_N(\alpha_1,\,\alpha_2,\,\alpha_3,\,\beta'_1,\,\beta_2\,;\,\gamma_1,\,\gamma'_2\,;\,x,\,y,\,z)$

$$=\sum_{m=0}^{\infty}\frac{(\beta_2, m)(\alpha_2+\alpha_3, m)}{(1, m)(\gamma_2, m)}y^m\times$$

$$\int_0^1 t^{\alpha_3-1}(1-t)^{\alpha_2+m-1}F_2(\beta_1\,;\,\alpha_1,\,\alpha_2+\alpha_3+m\,;\,\gamma_1,\,\gamma_2+m\,;\,x,\,tz)\,dt$$

valid if $R(\alpha_3) > 0$, $R(\gamma_2) > 0$ and $R(\alpha_2+\alpha_3) > 0$

Using the transformation [Bailey, *l.c.*]

$$F_2(\alpha\,;\,\beta,\,\beta'\,;\,\gamma,\,\gamma'\,;\,x,\,y) = (1-x)^{-\alpha}F_2\left(\alpha\,;\,\gamma-\beta,\,\beta'\,;\,\gamma,\,\gamma'\,;\,\frac{-x}{1-x},\,\frac{y}{1-x}\right) \quad (2)$$

$$F_N(\alpha_1, \alpha_2, \alpha_3, \beta'_1, \beta_2 ; \gamma_1, \gamma'_2 ; x, y, z)$$

$$= (1-x)^{-\beta} F_N\left(\gamma_1-\alpha_1, \alpha_2, \alpha_3 ; \beta'_1, \beta_2 ; \gamma_1, \gamma'_2 ; \frac{-x}{1-x}, y, \frac{z}{1-x}\right)$$

[Shanti Saran, *l.c.*, identity (5.13)]

Further substituting $v = (1-s)$, $w = st$, the integral becomes

$$\int_0^1\int_0^1\int_0^1 u^{\alpha_1-1}s^{\gamma_2-\alpha_2-1}t^{\alpha_3-1}(1-u)^{\gamma_1-\alpha_1-1}(1-s)^{\alpha_2-1}(1-t)^{\gamma_2-\alpha_2-\alpha_3-1}$$

$$\times (1-y+sy)^{-\beta_2}(1-ux-stz)^{-\beta_1}du\,ds\,dt$$

Using (1) again we obtain

$$\frac{\Gamma(\gamma_2-\alpha_2)\Gamma(\alpha_2)}{\Gamma(\gamma_2)} F_N = (1-y)^{-\beta_2}\int_0^1 s^{\gamma_2-\alpha_2-1}(1-s)^{\alpha_2-1}\left(1+\frac{sy}{1-y}\right)^{-\beta_2} \times$$

$$F_2(\beta_1 ; \alpha_1, \alpha_3 ; \gamma_1, \gamma_2-\alpha_2 ; x, sz)ds$$

Using the transformations of F_2 [Bailey, *l.c.*], as before, we obtain

$$F_N(\alpha_1, \alpha_2, \alpha_3 ; \beta'_1, \beta_2 ; \gamma_1, \gamma'_2 ; x, y, z)$$

$$= (1-x)^{-\beta_1}(1-y)^{-\beta_2} \times$$

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\beta_1, n+p)(\gamma_1-\alpha_1, n)(\alpha_3, p)}{(1, n)(1, p)(\gamma_1 n)(\gamma_2, p)}\, {}_2F_1\left[\beta_2, \gamma_2-\alpha_2+p ; \gamma_2+p ; \frac{-y}{y-1}\right] \times$$

$$\left(\frac{-x}{1-x}\right)^n\left(\frac{z}{1-x}\right)^p \qquad (3.1)$$

$$= (1-y)^{-\beta_2}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\beta_1, n+p)(\gamma_1-\alpha_1, n)(\gamma_2-\alpha_2-\alpha_3, p)}{(1, n)(1, p)(\gamma_1, n)(\gamma_2, p)} \times$$

$$F_1\left(\gamma_2-\alpha_2+p ; \beta_2, \beta_1+n+p ; \gamma_2+p ; \frac{-y}{1-y}, z\right)x^n(-z)^p \qquad (3.2)$$

$$= (1-y)^{-\beta_2}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(\beta_1, n+p)(\gamma_1-\alpha_1, n)(\gamma_2-\alpha_2-\alpha_3, p)}{(\gamma_1, n)(1, p)(\gamma_1, n)(\gamma_2, p)} \times$$

$$F_1\left(\gamma_2-\alpha_2+p ; \beta_2, \beta_1+n+p ; \gamma_2+p ; \frac{-y}{1-y}, \frac{z}{1-x}\right)\left(-\frac{x}{1-x}\right)^n\left(\frac{-z}{1-x}\right)^p \qquad (3.3)$$

From (3.2) we find that for $\gamma_2 = \alpha_2+\alpha_3$

$$F_N(\alpha_1, \alpha_2, \alpha_3 ; \beta'_1, \beta_2 ; \gamma_1, \alpha_2+\alpha'_3 : x, y, z) \quad \text{[Bailey, } l.c.\text{]}$$

$$= (1-y)^{-\beta_2} F_M\left(\alpha_1, \alpha'_3 ; \beta_2, \beta'_1 ; \gamma_1, \alpha_2+\alpha_3 ; x, \frac{-y}{1-y}, z\right)$$

$$= (1-y)^{-\beta_2} F_M\left(\gamma_1-\alpha_1, \alpha'_3 ; \beta'_1, \beta_2 ; \gamma_1, \alpha_2+\alpha_3 ; \frac{-x}{1-x}, \frac{-y}{1-y}, \frac{z}{1-x}\right)$$

from (3.3)

Also $\alpha_1 = 0$ in (3.2) leads to relation (12) of Appel *et al.*, *l.c.*, p. 35.

Again putting $w = 1-s$, $v = st$, the integral takes the form

$$\int_0^1\int_0^1\int_0^1 u^{\alpha_1-1}s^{\gamma_2-\alpha_3-1}t^{\alpha_2-1}(1-u)^{\gamma_1-\alpha_1-1}(1-s)^{\alpha_3-1}(1-t)^{\gamma_2-\alpha_2-\alpha_3-1}\times$$
$$(1-sty)^{-\beta_2}(1-ux-z+sz)^{-\beta_1}du\,ds\,dt$$

$$= (1-z)^{-\beta_1}\sum_{m=0}^{\infty}\frac{(\beta_2, m)}{(1, m)}y^m\int_0^1\int_0^1\int_0^1 u^{\alpha_1-1}s^{\gamma_2-\alpha_3+m-1}t^{\alpha_2+m-1}(1-u)^{\gamma_1-\alpha_1-1}\times$$
$$(1-s)^{\alpha_3-1}(1-t)^{\gamma_2-\alpha_2-\alpha_3-1}\left(1-\frac{ux}{1-z}+\frac{sz}{1-z}\right)^{-\beta_1}$$

valid if $|y| < 1$

Using the integral of F_2 *i.e.* (1) and evaluating the integral for t, we obtain

$$F_N(\alpha_1, \alpha_2, \alpha_3; \beta'_1, \beta_2; \gamma_1, \gamma'_2; x, y, z)$$
$$= (1-z)^{-\beta}\sum_{m=0}^{\infty}\frac{(\alpha_2, m)(\beta_2, m)y^m}{(1, m)(\gamma_2, m)}F_2\left(\beta; \sigma_1, \gamma_2-\alpha_3+m; \gamma_1, \gamma_2+m; \frac{x}{1-z}, \frac{-z}{1-z}\right) \qquad (3.2)$$

If $\gamma_2 = \alpha_2+\alpha_3$ we obtain the known result [Shanti Saran, *l.c*, identity (6.2)].

Again

$$(1-ux-wz)^p = (-)^p\sum_{m+n=0}^{m+n=p}\frac{(-p, m+n)}{(1, m)\,(1, n)}(1-ux)^m(1-wz)^p \qquad (8)$$

Writing this series for $(1-ux-wz)^p$ and using the hypergeometric functions ${}_2F_1$ and F_3 for their respective integrals we obtain

$$F_N[\alpha_1, \alpha_2, \alpha_3; -p', \beta_2; \gamma_1, \gamma'_2; x, y, z]$$
$$= (-)^p\sum_{m+n=0}^{m+n=p}\frac{(-p, m+n)}{(1, m)\,(1, n)}\,{}_2F_1(-m, \alpha_1; \gamma_1; x)\,F_3(\beta_2, -n, \alpha_2\,\alpha_3; \gamma_2; y, z) \qquad (3.5)$$

(i) $\beta_1 = 0$ leads to result (15) of Appell *et al.* [*l.c.*, p. 36]

4. Consider the integral

$$\iiint u^{\beta_1-1}v^{\beta_2-1}w^{\beta_3-1}(1-u-v-w)^{\gamma_1-\beta_1-\beta_2-\beta_3-1}(1-ux)^{-\alpha_1}(1-vy-wz)^{-\alpha_2}du\,dv\,dw$$

$$R(\gamma_1) > R(\beta_1+\beta_2+\beta_3) > 0 \text{ where } R(\beta_1) > 0,\ R(\beta_2) > 0,\ R(\beta_3) > 0$$
$$u \geq 0,\ v \geq 0,\ w \geq 0,\ u+v+w \leq 1.$$

Put $v = s(1-t)$, $w = st$, the integral becomes

$$\iiint_0^1 u^{\beta_1-1}s^{\beta_2+\beta_3-1}t^{\beta_3-1}(1-u-s)^{\gamma_1-\beta_1-\beta_2-\beta_3-1}(1-t)^{\beta_2-1}\times$$
$$(1-ux)^{-\alpha_1}(1-sy+sty-stz)^{-\alpha_2}du\,ds\,dt$$

Writing $(1-sy+sty-stz)^{-\alpha_2}$

$$\text{(i)} \qquad =\sum_{m=0}^{\infty}\frac{(\alpha_2,\ m)}{(1,\ m)}(z-y)^m s^m t^m (1-sy)^{-\alpha_2-m} \quad 1 \text{ valid if } \left|\frac{z-y}{1-y}\right|<1$$

$$\text{(ii)} \qquad =\sum_{m=0}^{\infty}\frac{(\alpha_2,\ m)}{(1,\ m)}s^m y^m[1-t(1-z/y)]^m \quad \text{valid if } \ |y|+|z|<1$$

using (i) in the integral and changing the order of integration and summation if

$\left|\frac{z-y}{1-y}\right|<1$ in $0\leq s\leq 1$, $0\leq t\leq\leq 1$, we get

$$\sum_{m=0}^{\infty}\frac{(\alpha_2,\ m)}{(1,\ m)}(z-y)^m\iiint_0^1 u^{\beta_1-1}s^{\beta_2+\beta_3+m-1}t^{\beta_3+m-1}\times$$
$$(1-u-s)^{\gamma_1-\beta_1-\beta_2-\beta_3-1}(1-t)^{\beta_2-1}(1-ux)^{-\alpha_1}(1-sy)^{-\alpha_2-m}\,du\,ds\,dt$$

Evaluating the integral for t and writing F_3 for its integral we obtain, finally,

$$F_S(\alpha_1,\ \alpha_2',\ \beta_1,\ \beta_2,\ \beta_3\ ;\ \gamma_1'\ ;\ x,\ y,\ z)$$
$$=\sum_{m=0}^{\infty}\frac{(\alpha_2,\ m)\ (\beta_3,\ m)}{(1,\ m)\ (\gamma_1,\ m)}\ (z-y)^m F_3(\alpha_1,\ \alpha_2+m,\ \beta_1,\ \beta_2+\beta_3+m\ ;\ \gamma_1+m\ ;\ x,\ y) \tag{4.1}$$

Expanding the series on the right hand side and putting $y=z$ we obtain

$$F_S(\alpha_1,\ \alpha'_2,\ \beta_1,\ \beta_2,\ \beta_3\ ;\ \gamma'_1,\ x,\ y,\ z)=F_3\ (\alpha_1,\ \alpha_2\ ;\ \beta_1,\ \beta_2+\beta_3\ ;\ x,\ y)$$

Again using (ii) and changing the order of integration and summation if $|y|+|z|<1$ and for writing $s=(1-u)p$, the integral becomes

$$\sum_{m=0}^{\infty}\frac{(\alpha_2,\ m)}{(1,\ m)}y^m\int_0^1\int_0^1\int_0^1 u^{\beta_1-1}t^{\beta_3-1}p^{\beta_2+\beta_3+m-1}(1-u)^{\gamma_1-\beta_1+m-1}(1-t)^{\beta_2-1}\times$$
$$(1-p)^{\gamma_1-\beta_1-\beta_2-\beta_3-1}(1-ux)^{-\alpha_1}\{1-t(1-z/y)\}^m\,du\,dt\,dp.$$

Evaluating the integral for p and writing ${}_2F_1$ for their respective integrals we obtain

$$F_S(\alpha_1,\ \alpha'_2,\ \beta_1,\ \beta_2,\ \beta_3\ ;\ \gamma'_1\ ;\ x,\ y,\ z)$$
$$=\sum_{m=0}^{\infty}\frac{(\alpha_2,\ m)\ (\beta_2+\beta_3,\ m)}{(1,\ m)\ (\gamma_1,\ m)}\,{}_2F_1\Big(\alpha_1,\ \beta_1\ ;\ \gamma_1+m\ ;\ x)\ {}_2F_1(-m,\ \beta_3\ ;\ \beta_2+\beta_3\ ;\ 1-\frac{z}{y}\Big)y^m \tag{4.2}$$

(1) $\alpha_1=0$ leads to result (8) of Appell *et al.*, [*l.c.*, p. 34]

Further applying Euler's transformation [Appell *et al.*, *l.c.*]

$${}_2F_1(a,\ b\ ;\ c\ ;\ z)=(1-z)^{-b}{}_2F_1\left(c-a,\ b\ ;\ c\ ;\ 1-\frac{z}{1-z}\right)$$

to the second hypergeometric function, we obtain

$$F_S(\alpha_1,\ \alpha'_2,\ \beta_1,\ \beta_2,\ \beta_3\ ;\ \gamma'_1\ ;\ x,\ y,\ z)$$
$$=z^{\beta_3}y^{-\beta_3}F_N(\beta_2,\ \alpha_1,\ \alpha_2\ ;\ \beta_1,\ \beta_2+\beta'_3\ ;\ \beta_2+\beta_3,\ \gamma'_1\ ;\ 1-y/z,\ x,\ y) \tag{4.3}$$

Two more results of some nature can be obtained by using the other Euler's transformations [Appell *et at. l.c.*]

Similarly applying the above transformation to the first and second hypergeometric functions and putting $\gamma_1 = \alpha_1 + \alpha_2$ we obtain

$$F_S(\gamma_1, \alpha'_2, \beta_1, \beta_2, \beta_3; \alpha_1 + \alpha'_2; x, y, z)$$

$$= (1-x)^{-\beta_1} z^{-\beta_2} y^{\beta_2} x \, F_N\left(\alpha'_2, \beta_3, \beta_1, \beta_2 + \beta'_3; \gamma'_1, \sigma_3 + \alpha_2; \frac{-x}{1-x}, y, 1 - \frac{y}{z}\right) \tag{4.4}$$

$\beta_1 = 0$ leads to result (9) of Appell *et al.*, [*l.c.*, p. 35]. Similar other transformations can be obtained by applying to both hypergeometric functions Euler's other transformations [Appell *et al.*, *l c.*,]. Thus making all possible combinations we get nine transformations of the same form.

Further let us substitute $u = 1 - s$, $v = st$ in the other integral for F_S namely

$$\iint u^{\alpha_1 - 1} v^{\alpha_2 - 1} (1-u-v)^{\gamma_1 - \alpha_1 - \alpha_2 - 1} (1-ux)^{-\beta_1} (1-vy)^{-\beta_2} (1-vz)^{-\beta_3} du\, dv$$

$$R(\gamma_1) > R(\alpha_1 + \alpha_2) > 0 \text{ where } R(\alpha_1) > 0, R(\alpha_2) > 0$$

we get $$\int_0^1 \int_0^1 s^{\gamma_1 - \alpha_1 - 1} t^{\alpha_2 - 1} (1-s)^{\alpha_1 - 1} (1-t)^{\gamma_1 - \alpha_1 - \alpha_2 - 1} (1 - x + sx)^{-\beta_1} (1 - sty)^{-\beta_2} (1 - stz)^{-\beta_3} ds\, dt$$

Using F_1 [Bailey, *l.c.*] for its integral and applying the wellknown transformation

$$F_1(\alpha; \beta, \beta'; \gamma, \gamma'; x, y)$$

$$= (1-x)^{-\beta} (1-y)^{-\beta'} F_1\left(\gamma - \alpha; \beta, \beta'; \gamma, \gamma'; \frac{-x}{1-x}, \frac{-y}{1-y}\right)$$

and using the hypergeometric function F_D [Appell *et al.*, *l.c.*] for integral ~~after~~ writing down the respective series of F_1 we get, finally.

$$F_S(\alpha_1, \alpha'_2, \beta_1, \beta_2, \beta_3, \gamma'_1, x, y, z)$$

$$= (1-x)^{-\beta_1} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\gamma_1 - \alpha_1 - \alpha_2, n+p)(\beta_2, n)(\beta_3, p)}{1, n)(1, p)(\gamma_1, n+p)} (-y)^n (-z)^p x \tag{4.5}$$

$$F_D\left((\gamma_1 - \alpha_1 + n + p; \beta_1, \beta_2 + n, \beta_3 + p; \gamma_1 + n + p; \frac{-x}{1-x}, y, z\right)$$

(i) $\gamma_1 = \alpha_1 + \alpha_2$, gives $F_S(\alpha_1, \alpha'_2, \beta_1, \beta_2, \beta_3; \alpha_1 + \alpha'_2; x, y, z)$

$$= (1-x)^{-\beta_1} F_D\left(\alpha_2; \beta_1, \beta_2, \beta_3, \alpha_1 + \alpha_2; \frac{-x}{1-x}, y, z\right)$$

(ii) $\alpha_1 = 0$, gives $F_1(\ _2; \beta_2, \beta_3; \gamma_1; y, z)$

$$(1-y)^{-\beta_2} (1-z)^{-\beta_3} F_1\left(\gamma_1 - \alpha_2; \beta_2, \beta_3; \gamma_1; \frac{-y}{1-y}, \frac{-z}{1-z}\right)$$

a wellknown transformation [Bailey, *l.c.*]

Similar other transformations follow by using the four transformations of F_I.

Further $$(1-vy-wz)^p = (-)^p \sum_{m+n=0}^{m+n=p} \frac{(-p,\, m+n)}{(1,\, m)(1,\, n)}(1-vy)^m(1-wz)^n.$$

Using this and writing F_B [Appell *et at.*, *l.c.*,] for its integral we finally obtain

$$F_S(\alpha_1, -p'.\ \beta_1, \beta_2, \beta_3, \gamma'_1;\ x, y, z)$$

$$= (-)^p \sum_{m+n=0}^{m+n=p} \frac{(-p,\, m+n)}{(1,\, m)\,(1,\, n)} F_B(\beta_1, \beta_2, \beta_3;\ \alpha_1, -m, -n;\ \gamma_1;\ x, y, z)$$

(i) $\alpha_1 = 0$ leads to result (14) of Appell *et al.* [*l.c.*, p. 36]

5. Consider the integral for F_T namely

$$\iint u^{\beta_1-1} v^{\beta_2-1} (1-u-v)^{\gamma_1-\beta_1-\beta_2-1} (1-vx)^{-\alpha_1} (1-uy-vz)^{-\alpha_2}\, du\, dv$$

$$R(\gamma_1) > R(\beta_1+\beta_2) > 0 \text{ where } R(\beta_1) > 0,\ R(\beta_2) > 0$$
$$u \geq 0,\ v \geq 0,\ u+v \leq 1.$$

Making the substitution $u = s(1-t)$, $v = st$, the integral takes the form

$$\int_0^1 \int_0^1 u^{\beta_1+\beta_2-1} t^{\beta_2-1} (1-s)^{\gamma_1-\beta_1-\beta_2-1} (1-t)^{\beta_1-1} (1-stx)^{-\alpha_1} (1-sy+sty-stz)^{-\alpha_2}\, ds\, dt$$

Writing $(1-sy+sty-stz)^{-\alpha_2}$

(i) $$= \sum_{m=0}^{\infty} \frac{(\alpha_2,\, m)}{(1,\, m)} s^m t^m (z-y)^m (1-sy)^{-\alpha_2-m}$$

valid if $\left|\frac{z-y}{1-y}\right| < 1$, in the domain of s and t

(ii) $$\sum_{m=0}^{\infty} \frac{(\alpha_2,\, m)}{(1,\, m)} s^m y^m \left[1 - t\left(1 - \frac{z}{y}\right)\right]^m$$

valid if $|y|+|z| < 1$, in the domain of s and t

using (i) and writing ${}_2F_1$ for its integral, after reversing the order of integration and summation under the condition $\left|\frac{z-y}{1-y}\right| < 1$, $0 \leq s \leq 1$, $0 \leq t \leq 1$, we get

$$\frac{\Gamma(\gamma_1-\beta_1-\beta_2)(\Gamma(\beta_1+\beta_2)}{\Gamma(\gamma_1)} F_T = \sum_{n=0}^{\infty} \frac{(\alpha_2,\, n)(\beta_2,\, n)}{(1,\, n)\,(\beta_1+\beta_2,\, n)} (z-y)^n \times$$

$$\int_0^1 s^{\beta_1+\beta_2+n-1} (1-s)^{\gamma_1-\beta_1-\beta_2-1} (1-sy)^{-\alpha_2-n}\ {}_2F_1(\alpha_1, \beta_2+n;\ \beta_1+\beta_2+n;\ sx)\, ds$$

$$R(\gamma_1) > R(\beta_1+\beta_2) > 0 \text{ where } R(\beta_1) > 0,\ R(\beta_2) > 0$$

Again applying Euler's first transformation [Appell *et al.*, *l.c.*] and writing its corresponding series we get

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha_1, m)(\alpha_2, n)(\beta_1, m)(\beta_2, n)}{(1, m)(1, n)(\beta_1+\beta_2, m+n)}(z-y)^n(-x)^m \times$$

$$\int_0^1 s^{\beta_1+\beta_2+m+n-1}(1-s)^{\gamma_1-\beta_1-\beta_2-1}(1-s)^{-\alpha_1-m}(1-sy)^{-\alpha_2-n}ds$$

Writing F_1 for its integral we finally get

$$F_T(\alpha_1, \alpha'_2, \beta'_1, \beta_2; \gamma'_1; x, y, z)$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha_1, m)(\alpha_2, n)(\beta_1, m)(\beta_2, n)}{(1, m)(1, n)(\gamma_1, m+n)}(-x)^m(z-y)^n x \tag{5.1}$$

$$F_1(\beta_1+\beta_2+m+n; \alpha_1+m, \alpha_2+n; \gamma_1+m+n; x, y)$$

Deductions : –

(i) $\alpha_1=0$ leads to result (7) of Appell *et al.*, [*l.c.*, p. 85].

(ii) $F_T(\alpha_1, \alpha'_2\, \beta'_1, -\beta_1; \gamma'_1, x, y, z)$

$$= F_3(\alpha_1, \alpha_2; \beta_1, -\beta_1, \gamma_1; -x, z-y)$$

(iii) $F_T(\alpha_1, \alpha'_2, \beta'_1, \beta_2; \beta_1+\beta_2; x, y, z)$

$$=(1-x)^{-\alpha_1}(1-y)^{-\alpha_2}F_3\left(\alpha_1, \alpha_2, ; \beta_1, \beta_2; \frac{x}{1-x}, \frac{z-y}{1-y}\right)$$

Writing (ii) expression for $(1-sy+sty-stz)^{-\alpha_2}$ in the series and writing ${}_2F_1$ for its corresponding integral, after changing the order of integration and summation, we obtain

$$\frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\beta_1+\beta_2)}F_T(\alpha_1, \alpha'_2, \beta'_1, \beta_2, \gamma'_1; x, y, z)$$

$$\sum_{m=0}^{\infty}\frac{(\alpha_2, m)(\beta_1+\beta_2, m)}{(1, m)(\gamma_1, m)}y^m \times$$

$$\int_0^1 u^{\beta_2-1}(1-u)^{\beta_1-1}\left[1-u\left(1-\frac{z}{y}\right)\right]^m {}_2F_1(\alpha_1, \beta_1+\beta_2+m; \gamma_1+m; ux)du$$

Applying Euler's transformation [Appell *et al.*, *l.c.*]

$${}_2F_1(a, b; c; z) = (1-z)^{-a}{}_2F_1\left(a, c-b, c; \frac{-z}{1-z}\right)$$

and writing the corresponding series and changing the order of integration and summation we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha_1, m)(\alpha_2, n)(\gamma_1-\beta_1-\beta_2, m)(\beta_1+\beta_2, n)}{(1, m)(1, n)(\gamma_1, m+n)}(-x)^m y^n x \times$$

$$\int_0^1 u^{\beta_2+m-1}(1-u)^{\beta_1-1}\left[1-u\left(1-\frac{z}{y}\right)\right]^m(1-ux)^{-\alpha_1-m}\,du$$

$$R(\beta_1)>0,\ R(\beta_2)>0 \text{ and } R(\beta_1+\beta_2)>0$$

Writing the hypergeometric function [Bailey, *l.c.*] for F_1 for its corresponding integral, we get finally,

$$F_T(\sigma_1,\alpha'_2,\beta'_1,\beta_2,\gamma'_1;x,y,z)$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha_1,m)(\alpha_2,n)(\gamma_1-\beta_1-\beta_2,m)(\beta_1+\beta_2,n)}{(1,m)(1,n)(\gamma_1,m+n)}(-x)^m y^n\times \tag{5.2}$$

$$F_1(\beta_2+m;\alpha_1+m,-n;\beta_1+\beta_2,m;x,1-z/y)$$

Making the transformation [Bailey, *l.c.*]

$$F_1(\alpha,\beta,\beta';\gamma;x,y)$$

$$=(1-x)^{-\beta}(1-y)^{-\beta'}F_1\left(\gamma-\alpha;\beta,\beta';\gamma;\frac{-x}{1-x},\frac{-y}{1-y}\right)$$

we get $F_T(\alpha_1,\alpha'_2,\beta'_1,\beta_2,\gamma'_1;x,y,z)$

$$=(1-x)^{-\alpha_1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha_1,m)(\alpha_2,n)(\gamma_1-\beta_1-\beta_2,m)(\beta_1+\beta_2,n)}{(1,m)(1,n)(\gamma_1,m+n)}(-x)^m z^n$$

$$F_1\left(\beta_1;\alpha_1+m,-n;\beta_1+\beta_2-m;\frac{-x}{1-x},1-\frac{z}{y}\right)$$

Lastly if we put $u=1-s$, $v=st$, the integral for F_T takes the form .

$$\int_0^1\int_0^1 s^{\gamma_1-\beta_1-1}t^{\beta_2-1}(1-s)^{\beta_1-1}(1-t)^{\gamma_1-\beta_1-\beta_2-1}(1-stx)^{-\alpha_1}(1-y+sy-stz)^{-\alpha_2}\,ds\,dt$$

$$(1-y+sy-stz)^{-\alpha_2}$$

$$=\sum_{m=0}^{\infty}\frac{(\alpha_2,m)}{(1,m)}s^m t^m z^m(1-y+sy)^{-\alpha_2-m}$$

$$\text{valid if } 2|y|+|z|<1 \text{ in } 0\le s\le 1,\ 0\le t\le 1$$

Thus applying this expansion and writing ${}_2F_1$ for the corresponding integral we get

$$\frac{\Gamma(\beta_1)\Gamma(\gamma_1-\beta_1)}{\Gamma(\gamma_1)}$$

$$=(1-y)^{-\alpha_1}\sum_{m=0}^{\infty}\frac{(\alpha_2,m)(\beta_2,m)}{(\gamma_1-\beta_1,m)(1,m)}\left(\frac{z}{1-y}\right)^m\times$$

$$\int_0^1 s^{\gamma_1-\beta_1+m-1}(1-s)^{\beta_1-1}\left(1+\frac{sy}{1-y}\right)^{-\alpha_2-m}{}_2F_1(\alpha_1,\beta_2+m;\gamma_1-\beta_1+m;sx)\,ds$$

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Applying Euler's first transformation [Appell *et al.*, *l.c.*] and writing the series, we obtain after writing F_1 for the corresponding integral

$$F_T(\alpha_1, \alpha'_2, \beta'_1, \beta_2 ; \gamma'_1 ; x, y, z)$$

$$= (1-y)^{-\alpha_2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha_1, m)(\alpha_2, n)(\gamma_1-\beta_1-\beta_2, m)(\beta_2, n)}{(1, m)(1, n)(\gamma_1, m+n)} (-x)^m \left(\frac{z}{1-y}\right)^n \times$$

$$F_1(\gamma_1-\beta_1+m+n, \alpha_1+m, \alpha_2+n, \gamma_1+m+n ; x, -y/1-y)$$

Two more transformations of similar nature can be obtained by applying the other two Euler's transformations [Appell *et al.*, *l.c.*]

Deduction :

(i) $F_T(\alpha_1, \alpha'_2, \beta'_1, \beta_2 ; \beta'_1 ; x, y, z)$

$$= (1-y)^{-\alpha_2} F_3\left(\alpha_1, \alpha_2, -\beta_2, \beta_2 ; \beta_1 ; -x, \frac{z}{1-y}\right) \qquad (5.3a)$$

(ii) $F_T(\alpha_1, \alpha'_2, \beta'_1, \beta_2 ; \beta_1+\beta_2 ; x, y, z)$

$$= (1-y)^{-\alpha_2}\, {}_2F_1\left(\alpha_2, \beta_2 ; \beta_1+\beta_2 ; \frac{z-y}{1-y}\right) \qquad (5.3b)$$

6. The integral for F_M is given by

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\gamma_2-\alpha_1)\Gamma(\gamma_2-\alpha_2)}{\Gamma(\gamma_1)\Gamma(\gamma_2)} F_M$$

$$= \int_0^1 \int_0^1 u^{\alpha_1-1} v^{\alpha_2-1} (1-u)^{\gamma_1-\alpha_1-1} (1-v)^{\gamma_2-\alpha_2-1} (1-vy)^{-\beta_2} (1-un-vz)^{-\beta_1} du\,dv$$

$$R(\gamma_1) > R(\alpha_1) > 0, \; R(\gamma_2) \geq R(\alpha_2) > 0.$$

Making the substitutions

(i) $u = 1-u', \; v = v'$

(ii) $u = u', \; v = 1-v'$

(iii) $u = 1-u', \; v = 1-v'$

we deduce the following transformations

$$F_M(\alpha_1, \alpha'_2, \beta'_1, \beta_2 ; \gamma_1, \gamma'_2 ; x, y, z)$$

$$= (1-x)^{-\beta_1} F_M\left(\gamma_1-\alpha_1, \alpha'_1, \beta_1, \beta_2 ; \gamma_1, \gamma_2 ; \frac{-x}{1-x}, y, \frac{z}{1-x}\right)$$

$$= (1-y)^{-\beta_2} (1-z)^{-\beta_1} F_M\left(\alpha_1, \gamma_2-\alpha'_2, \beta_1, \beta_2 ; \frac{x}{1-z}, \frac{-y}{1-y}, \frac{-z}{1-z}\right)$$

$$= (1-x-z)^{-\beta_1} (1-y)^{-\beta_2} F_M\left(\gamma_1-\alpha_1, \gamma_2-\alpha'_2 ; \beta_1, \beta_2 ; \frac{-x}{1-x-z}, \frac{-y}{1-y}, \frac{-z}{1-x-z}\right)$$

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STRONG CONVERGENCE OF ROBBINS AND MONRO AND KIEFER AND WOLFOWITZ PROCESSES[*]

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The Robbins-Monro stochastic approximation process determines the root of the equation $m(x) = 0$, where $m(x) = E(y \mid x)$ is a regression function. The terminology used here is the same as in Robbins and Monro (1952).

1. We assume the following conditions

$$\text{(1a)} \quad \limsup \left| \frac{m(x)-\alpha}{x-\theta} \right| < K \text{ as } |x| \to \infty, \qquad \text{(1b)} \quad \sigma_x^2 = \int_{-\infty}^{\infty} [y-m(x)]^2 dH(y \mid x) \leq \sigma^2$$

where $H(y \mid x)$ *is the conditional distribution of* y *when* x *is given.* This is more general than Wolfowitz's condition as it does not require $m(x)$ to be bounded as $|x| \to \infty$.

$$\text{(2)} \qquad \left.\begin{array}{lll} m(x) \leq \alpha - \pi(\delta) & \text{when} & x < \theta - \delta \\ m(x) \geq \alpha + \pi(\delta) & \ldots & x > \theta + \delta \end{array}\right\}$$

where $\pi(\delta) > 0$ *for all* δ.

This implies that $\pi(\delta)$ is non-decreasing function of δ.

(3) Σa_n *is divergent and* Σa_n^2 *is convergent.*

These conditions allow any curve $m(x)$ which is increasing in the neighbourhood of θ, though it need not be continuous at this point and such that beyond this neighborhood $m(x)$ differs from α by at least ϵ, and $m(x)-\alpha$ does not change too rapidly to $x-\theta$ as $|x-\theta|$ increases.

We shall show that *under the above conditions, the Robbins Morno stochastic approximation process*

$$x_{n+1} = x_n - a_n(y_n - \alpha)$$

converges in the strong sense to θ *where* $m(\theta) = \alpha$

i.e., $$\lim x_n = \theta \quad \textit{with probability one.}$$

Proof. Let
$$\left| \frac{m(x)-\alpha}{x-\theta} \right| < K_1 (K_1 \geq K) \text{ when } |x - \theta| > \tfrac{1}{2}c \tag{1.1}$$

we then have

$$x_{n+1} - \theta = \{x_n - \theta - a_n[m(x_n) - \alpha]\} - a_n[y_n - m(x_n)] \tag{1.2}$$

i.e., $$E\{|x_{n+1} - \theta|^2 \mid x_n\} = \{x_n - \theta - a_n[m(x_n) - \alpha]\}^2 + a_n^2 \sigma_{xn}^2.$$

[+] Read in the spring meeting of the Institute of Mathematical Statistics, held in Cornell. N. Y. in 1954.

From (3) Σa_n^2 is convergent and thus $\lim a_n = 0$ and for sufficiently large $n > n_0$

$$a_n K_1 < 1$$

and thus $$a_n[m(x_n) - 1] < |x_n - \theta| \text{ when } |x_n - \theta| > \tfrac{1}{2}\varepsilon.$$

Also $m(x_n) - \alpha$ has the same sign as $x_n - \theta$ and thus

$$\{(x_n - \theta) - a_n[m(x_n) - \alpha]\}^2 < (x_n - \theta)^2.$$

When $|x_n - \theta| \leq \frac{1}{2}\varepsilon$ we chose $n > n_0$ so that for $n > n_1$

$$a_n[m(x_n) - \alpha] < \tfrac{1}{2}\varepsilon$$

and thus in this case

$$\{(x_n - \theta) - a_n[m(x_n) - \alpha]\}^2 \leq \varepsilon^2.$$

We have thus

$$E\{(x_{n+1} - \theta)^2 \mid x_n\} \leq \text{Max}\,\{|x_n - \theta|^2, \varepsilon^2\} + a_n^2\sigma_{x_n}^2.$$

Let $$Z_n^2 = \text{Max}\,\{|x_n - \theta|^2, \varepsilon^2\}$$

then $$E\{Z^2{}_{n+1} \mid Z^2{}_n\} \leq Z^2{}_n + a^2{}_n\sigma^2{}_{x_n}.$$

Thus $$E\{Z^2{}_{n+1} \mid Z^2{}_{n_1}\} \leq Z^2{}_{n_1} + \sum_{j=n_1}^{n} a^2{}_j E(\sigma^2{}_{x_j} \mid x_{n_1})$$

$$\leq Z^2{}_{n_1} + \sum_{j=n_1}^{n} a^2{}_j\sigma^2.$$

$$\breve{x}_{n+1} = Z^2{}_{n_1} + \sigma^2 \sum_{j=n_1}^{n} a^2{}_j - Z^2{}_{n+1}$$

then $$E\{\breve{x}_{n+1} \mid \breve{x}_n\} = Z^2{}_{n_1} + \sigma^2 \sum_{j=n_1}^{n} a^2{}_j - E(Z^2{}_{n+1} \mid Z^2{}_n)$$

$$\geq Z^2{}_{n_1} + \sigma^2 \sum_{j=n_1}^{n} a^2{}_j - Z^2{}_n - a^2{}_n\sigma^2{}_{x_n}$$

$$\geq Z^2{}_{n_1} + \sigma^2 \sum_{j=n_1}^{n-1} a^2{}_j - Z^2{}_n$$

$$= \breve{x}_n.$$

Hence $\{\breve{x}_n\}_{n>n_1}$ forms a semimartingale with

$$E\{|\breve{x}_n| \mid Z_{n_1}\} \leq Z^2{}_{n_1} + \sum_{j=n_1}^{n-1} a^2{}_j\sigma^2 + E\{Z^2{}_n \mid Z_1\}$$

$$\leq 2\{Z^2{}_{n_1} + \sigma^2 \sum_{j=n_1}^{n-1} a^2{}_j$$

Hence from the convergence property of semimartingales (See Doob, 19.3, theorem 4.15, p. 324).

$$\lim x_n \text{ ezists with probability one.}$$

Hence $\lim z_n^2 = z^2$ exists as a random variable with probability one. This implies of course the convergence of $|Z_n|$ i.e., $\lim |Z_n| = |Z|$ with probability one. We show now show that $Z = 0$ with probability one

2. From (1.2)

$Z_{n+1} = \{Z_n - a_n[m(x_n) - \alpha]\} - a_n\xi_n$ where $Z_n = x_n - \theta$ and ξ_n is a random variable with mean zero and independent of $\xi_1, \xi_2, \ldots \xi_{n-1}$

Let now $$2\varepsilon - \tfrac{1}{4}\varepsilon < z_n < C \tag{2.1}$$

From condition (2) $$|m(x_n) - \alpha| > \pi(\varepsilon) \tag{2.2}$$

and from condition (1) $$|m(x_n) - \alpha| < K_1 |z_n|. \tag{2.3}$$

We now chose n so large that

$$a_n\pi(\varepsilon) > \varepsilon \text{ and } a_nK_1 < \varepsilon < \tfrac{1}{2}$$

since $m(x_n) - \alpha$ has the same sign as Z_n, we have

$$\begin{aligned} Z_{n+1} &\leq Z_n - a_n\pi(\varepsilon) - a_n\xi_n \\ Z_{n+1} &\geq Z_n - a_nK_1Z_n - a_n\xi_n \\ &= (1 - a_nK_1)Z_n - a_n\xi_n. \end{aligned} \tag{2.4}$$

Let now $$E_1:\quad Z_{n+1} \geq 5\varepsilon/4$$
$$F_1:\quad Z_{n+1} < -5\varepsilon/4.$$

From inequalities (2.3) and (2.4), these are implied by E'_1 and F'_1 respectively

$$E'_1:\quad a_n\xi_n \leq C + 5\varepsilon/4 - a_n\pi(\varepsilon)$$
$$F'_1:\quad a_n\xi_n \geq 5\varepsilon/8 + (1 + a_nk_1)5\varepsilon/4 \geq 2\varepsilon$$

when E_1 holds we have the inequalities

$$\begin{aligned} Z_{n+2} &\leq Z_n - a_n\pi(\varepsilon) - a_{n+1}\pi(\varepsilon) - a_n\xi_n - a_{n+1}\xi_{n+1}, \\ Z_{n+2} &\geq (1 - a_{n+1}K_1)Z_{n+1} - a_{n+1}\xi_{n+1}. \end{aligned} \tag{2.5}$$

Hence $E_2:\quad Z_{n+2} > 5\varepsilon/4$ and $F_2:\quad Z_{n+2} < -5\varepsilon/4$

are implied by

and $$E'_2:\quad a_n\xi_n + a_{n+1}\xi_{n+1} \leq C + 5\varepsilon/2 - \pi(\varepsilon)(a_n + a_{n+1})$$
$$F'_2:\quad a_{n+1}\xi_{n+1} \geq 2\varepsilon$$

We thus have the sequence of sets,

$$E'_i:\quad \sum_{n=0}^{i-1} a_{n+r}\xi_{n+r} \leq C + 5\varepsilon/4 - \pi(\varepsilon)\sum_{n=0}^{i-1} a_{n+1}$$

$$F'_i:\quad a_{n+i-1}\xi_{n+i-1} \geq 2\varepsilon$$

so that
$$E'_1 \cap \ldots \cap E'_i \cup E'_{i+1} \supset E_1 \cap E_2 \ldots \cap E_i \cap E_{i-1}$$
$$E'_1 \cap \ldots \cap E'_i \cap F'_{i+1} \supset E_1 \cap E_2 \ldots \cap E_i \cap F_i \tag{2.7}$$

where
$$E_i: \quad Z_{n+i} \geq 5\varepsilon/4 \text{ and } F_i: \quad Z_{n+i} \leq -5\varepsilon/4.$$

Now
$$\text{Prob}\left\{\bigcup_{i=1}^{\infty}[E'_1 \cap \ldots \cap E'_i \cap F'_{i+1}]\right\}$$
$$\leq \text{Prob}\left\{\bigcup_{i=1}^{\infty} F'_i\right\} + \text{Prob}\left\{\bigcap_{i=1}^{\infty} E'_i\right\}. \tag{2.8}$$

From
$$\text{Prob}\,\{a_j\xi_j \geq 2\varepsilon\} < \frac{a^2_j\sigma^2}{4\varepsilon^2}$$

$$\text{Prob}\left\{\bigcup_{i=n_1}^{\infty} F'_i\right\} \leq \frac{\sigma^2}{4\varepsilon^2}\sum_{j=n_1}^{\infty} a^2_j < \tfrac{1}{2}\delta \text{ when } n_1 \text{ is large.} \tag{2.9}$$

Consider now the probability of all the inequalities E'_n that is

$$\sum_{j=n_1}^{n_1+r-1} a_j\xi_j \leq C + \tfrac{1}{4}5\varepsilon - \pi(\varepsilon)\sum_{j=n_1}^{n_1+r-1} a_j$$

. Since Σa_j is divergent, we can make $\sum_{j=n_1}^{n_1+r-1} a_j > \left(C + \frac{9\varepsilon}{4}\right)/\pi(\varepsilon)$

for any n_1, by chosing r sufficietly large and we have

$$\sum_{j=n_1}^{n_1+r-1} a_j\xi_j < -\varepsilon \tag{2.10}$$

we can chose n_1 large so that probability of (2.10) is less then $\frac{1}{2}\delta$.

Thus from (2.7) (2.8), (2.9) and (2.10)

$$\underset{j=n_1,\, n_1+1, \ldots \quad \ldots\ldots}{\text{Prob}\left\{|Z_j| > \frac{5\varepsilon}{4} \,\middle|\, \frac{7\varepsilon}{4} \leq Z_{n_1} \leq C\right\}} < \delta,$$

$$i.e., \quad \text{Prob}\,\{(|Z_j| - |Z_{n_1}|) \geq \tfrac{1}{2}\varepsilon \,|\, 7\varepsilon/4 < Z_{n_1} \leq C\} > 1 - \delta$$

for at least one $j = n_1 + 1,\ n_1 + 2, \ldots\ldots.$

Let
$$\text{Prob}\,\{C > Z_{n_1} > 7\varepsilon/4\} = \beta_{n_1};$$

then $\text{Prob}\,\{||Z_j| - |Z_{n_1}|| < \frac{1}{2}\varepsilon$ for at least one $j = n_1 + 1,\ n_1 + 2, \ldots\} \geq \beta_{n_1}(1 - \delta)$

But as $|Z_n|$ converges strongly, we have for sufficiently large n,

$$\text{Prob}\,\{||Z_j| - |Z_n|| \geq \tfrac{1}{2}\varepsilon, \text{ for at least one } j = n+1,\ n+2, \ldots\} < \delta'.$$

Thus
$$\beta_{n_1} \leq \frac{\delta'}{1-\delta}$$

As δ and δ' are arbitarary it follows that $\lim \beta_n = 0$

Similarly if we had $-7\varepsilon/4 \geq Z_n \geq -C$ we would have

$$\lim \operatorname{Prob}\{-7\epsilon/4 > Z_n > C\} = 0$$

As C and ϵ are arbitrary, we have

$$\lim Z_n = 0,$$

Wolfowitz (1952) has given an example in which condition (1*b*) does not hold and shown that the process is not convergent. In another example Wolfowitz (1952) has shown that the condition (2) does not hold and the process is not convergent. An example given below shows that if condition (1*a*) does not hold then the process may not converge. It may observed that condition (1*a*) is more general that Wolfowitz's condition.

Example. Let $H(y \mid x) = 1$ when $y \geq x$

$= 0$ $\quad y < x$;

Let $x_1 = 1$ $\quad m(x_1) = 3$

$x_2 = -2$ $\quad m(x_2) = -5$

$x_n = (-1)^{n+1}(n+1)$ $\quad m(x_n) = (-1)^{n+1}(2n+1)$

In this case obviously the process cannot converge and of course condition (1*a*) does not hold. While the other conditions (1*b*) and (2) hold.

3. Stochastic estimation of the maximum of a regression function.

Keifer and Wolfowitz (1952) have extended the method of Robbins and Monro (1951) to the estimation of the maximum of a regression function. We shall show that the Keifer and Wolfowitz process (which we shall refer to as (KW)—process) is strongly convergent under conditions stated below, which are less restrictive than conditions stated in (KW).

We assume that the conditions below are satisfied

$$m(x) = \int_{-\infty}^{\infty} y\,dH(y \mid x)$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} [y - m(x)]^2 dH(y \mid x) \leq \sigma^2 < \infty \tag{3.1}$$

$m(x)$ is strictly increasing $x < \theta$ and strictly decreasing for $x \geq \theta$, (3.2)
where θ is the maximum of $m(x)$.

$\{a_n\}$ and $\{c_n\}$ are sequences such that

$$c_n \to 0 \text{ as } n \to \infty \tag{3.3}$$

$$\Sigma a_n \text{ is divergent} \tag{3.4}$$

$$\Sigma(a_n^2/c_n^2) \text{ is convergent} \tag{3.5}$$

There exist ϱ and R such that

$$|x' - x''| < \varrho \text{ implies } |m(x') - m(x'')| \leq R. \tag{3.6}$$

For every $\delta > 0$ *there exists a positive* $\pi(\delta)$

such that
$$|x-\theta| > \delta \text{ implies } \inf_{\frac{1}{2}\delta > \epsilon > 0} \left| \frac{m(x+\epsilon)-m(x-\epsilon)}{\epsilon} \right| < \pi(\delta) \tag{3.7}$$

Wolfowitz and Keifer assume two more conditions which are not needed here.

$$\Sigma c_n a_n \text{ is convergent.} \tag{3.8}$$

$$|x'-\theta|+|x''-\theta| < \beta \text{ implies } |m(x')-m(x'')| > B|x'-x''| \tag{3.9}$$

for some β *and* B, *which is a Lipschitz condition.*

The (KW) process is defined by

$$x_{n+1} = x_n + \frac{a_n}{c_n}\{y_{2n} - y_{2n-1}\} \tag{3.10}$$

where y_{2n} has mean $m(x_n+c_n)$ and y_{2n-1} has mean $m(x_n-c_n)$ but otherwise independent of x_n.

We may write

$$x_{n+1} - \theta = x_n - \theta + \frac{a_n}{c_n}\{m(x_n+c_n) - m(x_n-c_n)\} + \frac{a_n}{c_n}\{\xi_{2n} - \xi_{2n-1}\}. \tag{3.11}$$

where
$$\xi_{2n} = y_{2n} - m(x_n+c_n)$$
$$\xi_{2n-1} = y_{2n-1} - m(x_n-c_n)$$

From condition (1), the variances of ξ_{2n} and ξ_{2n-1} are bounded and so the variance of $\xi_n = \xi_{2n} - \xi_{2n-1}$ is bounded, *i.e.*, Var $(\xi_n) < 2\sigma^2$. We now consider the conditional expectation of $(x_{n+1}-\theta)^2$ when x_n is held fixed.

$$E\{|x_{n+1}-\theta|^2 \mid x_n\} = \{(x_n-\theta) + \frac{a_n}{c_n}[m(x_n+c_n) - m(x_n-c_n)]\}^2 + \frac{a_n^2}{c_n^2}\sigma_n^2. \tag{3.12}$$

as ξ_n is independent of x_n.

From condition (3) and (6), for sufficiently large n we have

$$c_n < \tfrac{1}{2}\epsilon < \varrho, \ |m(x_n+c_n) - m(x_n-c_n)| < R \text{ for } n > n_0$$

Let $\quad n > n_1 > n_0 \quad$ so that $\frac{a_n}{c_n} < \frac{\epsilon}{2R}$ which is possible from condition (5).

Thus $\frac{a_n}{c_n}|m(x_n+c_n) - m(x_n-c_n)| < \frac{1}{2}\epsilon$ when $n < n_1$ when $|x_n - \epsilon| < \frac{1}{2}\epsilon$, $x_n + c_n$ and $x_u - c_n$ lie on the same side of θ and thus the difference

$$m(x_n+c_n) - m(x_n-c_n)$$

has opposite sign to that of $x_n - \theta$

Thus in this case

$$\left\{(x_n-\theta)+\frac{a_n}{c_n}[m(x_n+c_n)-m(x_n-c_n)]\right\}^2 \leq (x_u-\theta)^2$$

when $|x_n-\theta| \geq \frac{1}{2}\varepsilon$

Also when $|x_n-\theta| < \frac{1}{2}\varepsilon$

we have

$$\left[(x_n-\theta)+\frac{a_n}{c_n}\{m(x_n+c_n)-m(x_n-c_n)\}\right]^2 \leq \varepsilon^2.$$

Thus we have

$$E\{|x_{n+1}-\theta|^2/x_n\} \leq \text{Max}\{|x_n-\theta|^2,\ \varepsilon^2\}+\frac{a_n^2}{c_n^2}\sigma^2$$

Let

$$Z_n^2 = \text{Max}\{|x_n-\theta|^2,\ \varepsilon^2\}$$

then

$$E[Z^2_{n+1}/Z_n^2] \leq Z_n^2+\frac{a_n^2}{c_n^2}\sigma^2 \tag{3.13}$$

from which

$$E(Z_n^2) \leq Z^2_{n_1}+\sigma^2\sum_{r=n_1}^{n-1}(a_r^2/c_r^2)$$

let

$$\overleftarrow{x} = \sigma^2\sum_{r=n_1}^{n-1}(a_r^2/c_r^2)+Z^4_{n_1}-Z^2_{n},$$

then

$$E\{\overleftarrow{x}_{n+1}/\overleftarrow{x}_n\} = Z^2_{n_1}+\sigma^2\sum_{r=n_1}^{n}(a^2_r/c^2_r)-(a^2_n/c^2_n)\sigma^2_{v_n}-Z^2_n$$

$$\geq Z^2_{n_1}+\sigma^2\sum_{r=n_1}^{n-1}(a^2_r/c^2_r)-Z^2_n$$

$$\geq \overleftarrow{x}_n, \tag{3.14}$$

Hence $\{\overleftarrow{x}_n\}$ is a semimartingale for which $E(|\overleftarrow{x}_n|$ is bounded and thus from the semi-martingale theorem of Doob (1953) $\lim \overleftarrow{x}_n$ exists with probability one, and thus

$$\lim Z_n^2 = Z^2 \tag{3.15}$$

exists with probability one as a random variable.

This implies that

$$\lim|Z_n| = |Z| \text{ with probability one}$$

We shall now show that $|Z| \leq 2\varepsilon$ with probability one so that $\lim Z_n = 0$ with probability one, since ε is arbitrary.

We can use the same proof as in section 2 by observing that instead of (2.2) we now have

$$\frac{m(x_n+c_n)-m(x_n-c_n)}{c_n} > \pi(\delta)$$

with holds for $|Z_n| > \delta$ and $c_n < \frac{1}{2}\delta$

i.e., for sufficiently large n, while for (2.3) we have

$$| m(x_n+c_n)-m(x_n-c_n) | < R < K_1 | Z_n |$$

for sufficiently large n, and $| Z_n | > \varepsilon$

where $$K_1 = R/\varepsilon$$

Since Σa_n is divergent and $\Sigma(a_n^2/c_n^2)$ and so also Σa_n^2 is convergent the proof in section 2 still holds in this case with trivial modifications.

Hence the theorem.

After this paper was written two papers were published in the *Annals of Mathematical Statitics*, June 1954, on this problem. The proceedure in these are quite different from the present one.

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ON (ϕ)-CONVERGENT INTEGRALS AND THEIR APPLICATION TO MATHEMATICAL PHYSICS

By

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Introduction. In this paper we study in a generalized manner certain properties of infinite integrals which oscillate in the ordinary sense. These propesties are similar to those studied in previous papers by the author (Ghosh, 1947, 1948, 1949, *Cf.* also Hardy 1912*a*, 1949, p. 319). With the help of these integrals Rutherford scattering formula has been treated by Born's method and the result has been obtained without any special hypothesis unlike Wentzel (1927). The method has also been applied to the evaluation of a very important infinite integral used by Bethe (1930).

I. Mathematical Theory

We consider a function $\varphi(x)$ with the following proporties:

(*i*) $\varphi(x)$ is monotonic non-increasing in $[0, \infty)$,

(*ii*) $\varphi'(x)$ is continuous in $[0, \infty)$,

(*iii*) $\int_0^\infty \varphi'(x)dx = \lim_{x\to\infty} \varphi(x) - \varphi(0)$ is finite and non-zero and equal to K (say).

Then we can easily prove that if a function $f(x)$ integrable-R in $[0, \infty)$ tend to a limit l (finite or infinite) as x tends to infinity then

$$\frac{1}{\lambda K}\int_0^\infty \varphi'(t/\lambda)f(t)dt$$

tends to the limit l as λ tends to infinity.

Definition. *If*

$$\frac{1}{\lambda K}\int_0^\infty \varphi'(t/\lambda)f(t)dt$$

tend to a limit l *as* $\lambda\to\infty$, *we call this* l *the* (φ)-*limit of* $f(x)$ *as* $x\to\infty$.

In the above definition we have taken the integral from 0, but this does not constitute any real restriction on the generality of the method.

Examples. (*i*) If $\left.\begin{aligned}\varphi(t) &= (1-t)^\alpha, \text{ for } t\leq 1\\ &= 0, \qquad \text{for } t>1\end{aligned}\right\}\ (\alpha>1),$

we get the case of (C, α)-limit. [*cf.* Doetsh, (1920)].

(*ii*) If $\varphi(t) = e^{-t}$, we get the case of Abel-limit, (Hardy, 1904).

(*iii*) If $\varphi(t) = e^{-t^2}$, we get a case of much inportance in Mathematical Physics

Let us now consider the integral

$$F(x) = \int_c^x f(t)dt. \tag{1.2}$$

If (φ)-lim $F(x)$, as $x \to \infty$, exists, we say that the integral on the right of (1.2) is (φ)-convergent and we take this limit as the value of the infinite integral. We denote this integral by

$$(\varphi)\text{-}\int_c^\infty f(x)dx$$

In the case of $c = 0$, the integral takes an elegant form

$$(\varphi)\text{-}\int_0^\infty f(x)dx = \lim_{\lambda\to\infty} \frac{1}{\lambda K} \int_0^\infty \varphi'(x/\lambda) \int_0^x f(x)dx$$

$$= -\lim_{\lambda\to\infty} \frac{1}{K} \int_0^\infty \varphi(x/\lambda)f(x)dt$$

provided that the right hand side exists and

$$\varphi(x/\lambda) \int_0^x f(x)dt \to 0 \quad \text{as } x \to \infty, \quad (\lambda \geq 0).$$

If, in addition, we assume that $\lim\limits_{x\to\infty} \varphi(x) = 0$, we get

$$(\varphi)\text{-}\int_0^\infty f(x)dx = \lim_{\delta\to\infty} \frac{1}{\varphi(0)} \int_0^\infty \varphi(\delta x)f(x), \; (\delta = 1/\lambda). \tag{1.3}$$

In this form the integral has been called *summable*(φ) by Hardy (Hardy 1912*b*, p. 431). We are, however, dealing with a more general notion of limit and we use the word *convergent* instead of *summable*. We also note in this connection that our conditions on φ are somewhat different and in a way more general than those of Hardy (Hardy, 1912*b*, p. 431, also *cf*. Hobson, 1926, p. 317, where furter references are given)

Uniform convergence. Let us consider integrals containing a parameter a.

Let
$$F(\lambda, a) = \frac{1}{\lambda K} \int_0^\infty \varphi'\left(\frac{t}{\lambda}\right) \int_c^t f(t, a)dt \tag{1.3}$$

If now $F(\lambda, a)$ *converges uniformly in an interval* $A \leq a \leq B$ *as* $\lambda \to \infty$, *we say that the integral*

$$(\varphi)\text{-}\int_c^\infty f(t, a)dt \tag{1,4}$$

converges uniformly in the interval $A \leq a \leq B$.

Continuity with respect to a parameter. **Theorem I.** *Let*

(*i*) $f(x, a)$ *be continuous in the region* $c \leq x < \infty$, $A \leq a \leq B$;

(*ii*)
$$\frac{1}{\lambda K}\int_0^\infty \varphi'\left(\frac{l}{\lambda}\right)\int_c^x f(t, a)dt \qquad (I/\lambda > 0)$$

converge uniformly in the interval $A \leq a \leq B$;

(*iii*)
$$(\varphi)\text{-}\int_c^\infty f(x, a)dx$$

converge uniformly in $A \leq a \leq B$, *then last integral defines a function*

$$\psi(a) = (\varphi)\text{-}\int_c^\infty f(x, a)dx$$

continuons in the interal $A \leq a \leq B$.

Proof. We take any point a_0 in the interval $A \leq a \leq B$. As the integral is uniformly convergent,

$$\psi(a_0) = \frac{1}{\lambda' K}\int_0^\infty dx.\varphi'(x/\lambda')\int_c^x f(t, a_0) + R_{\lambda'}(a_0)$$

and
$$\psi(a_0+h) = \frac{1}{\lambda' K}\int_0^\infty dx.\varphi'(x/\lambda')\int_c^x f(t, a_0+h)dt + R_{\lambda'}(a_0+h)$$

where both $|R_{\lambda'}(a_0)| < \varepsilon$ and $|R_{\lambda'}(a_0+h)| < \varepsilon$ when $\lambda' > \Lambda(\epsilon)$. Hence

$$|\psi(a_0+h)-\psi(a_0)| \leq \left|\frac{1}{K\lambda'}\int_0^\infty dx.\varphi'(x/\lambda')\int_c^x [f(t, a_0+h)-f(t, a_0)]dt\right| + |R_{\lambda'}(a_0+h)| + |R_{\lambda'}(a_0)|.$$

We now keep λ' fixed. Then, since, by virtue of (*ii*)

$$\frac{1}{\lambda' K}\int_0^\infty dx.\varphi'(x/\lambda)\int_c^x f(t, a)dt$$

is a continuons function of a in the interval $A \leq a \leq B$, it follows that

$$\left|\frac{1}{\lambda' K}\int_0^\infty dx.\varphi'(t/\lambda)\int_c^x [f(t, a_0+h)-f(t, a_0)]dt\right| > \varepsilon \quad |h| \geq \delta(\varepsilon)$$

Hence
$$|\psi(a_0+h)-\psi(a_0)| < 3\varepsilon, \quad |h| < \delta(\varepsilon)$$

which proves the proposition.

Integration with respect to a parameter: Reversal of the order of integration.

Theorem 2. *Let*

(*i*) $f(x, a)$ *be continuous in the region* $c \leq x < \infty$, $A \leq a \leq B$;

(*ii*) $$\frac{1}{\lambda K}\int_0^\infty \varphi'(x/\lambda)\int_c^x f(t, a)dt, \quad (1/\lambda > 0)$$

converge uniformly in the interval $A \leq a \leq B$,

(*iii*) $$(\varphi)\text{-}\int_c^\infty f(x, a)dx$$

converge uniformly in the interval $A \leq a \leq B$:

$$\psi(a) = (\varphi)\text{-}\int_0^\infty f(x, a)dx,$$

then $$\int_{a_0}^{a_1}\psi(a)da = (\varphi)\text{-}\int_c^\infty dx\int_{a_0}^{a_1} f(x, a)da, \quad (A \leq a_0 \leq a_1 \leq B)$$

or $$\int_{a_0}^{a_1} da.(\varphi)\text{-}\int_c^\infty f(x, a)dx = (\varphi)\text{-}\int_c^\infty dx\int_{a_0}^{a_1} f(x, a)da$$

Proof. The function $\psi(a)$ is continuous by Theorem 1 and therefore integrable. Because of uniform convergence

$$\psi(a) = \frac{1}{\lambda' K}\int_0^\infty dx.\varphi'(x/\lambda')\int_c^x f(t, a)dt + R_{\lambda'}(a)$$

where $|R_{\lambda'}(a)| < \epsilon$, when $\lambda' > \Lambda(\epsilon)$

$$\int_{a_0}^{a_1}\psi(a)da = \frac{1}{\lambda' K}\int_{a_0}^{a_1} da\int_0^\infty dx.\varphi'(x/\lambda')\int_c^x f(t, a)dt + \int_{a_0}^{a_1} R_{\lambda'}(a)da$$

We keep λ' fixed. The order of integrations in the first integral on the right can be reversed, by virtue of (*ii*), so that

$$\int_{a_0}^{a_1} da.(\varphi)\text{-}\int_c^\infty f(x, a)dx - \frac{1}{\lambda' K}\int_c^\infty \varphi'(x/\lambda')\int_c^x dt\int_{a_0}^{a_1} f(t, a)da - \int_{a_0}^{a_1} R_{\lambda'}(a)da.$$

Hence $$\left|\int_{a_0}^{a_1} da\ (\varphi)\text{-}\int_c^\infty f(x, a)dx - \frac{1}{\lambda' K}\int_0^\infty \varphi'(x/\lambda')\int_c^x dt\int_{a_0}^{a_1} f(t, a)da\right| < \epsilon\,|a_1 - a_0|$$

for $\lambda' > \Lambda(\epsilon)$. Therefore

$$\int_{a_0}^{a_1} da.(\varphi)\text{-}\int_c^\infty f(x, a)dx = (\varphi)\text{-}\int_c^\infty dx\int_{a_0}^{a_1} f(x, a)da.$$

Differentiation with respect to a parameter. Theorem 3. *Let*

(*i*) $$(\varphi)\text{-}\int_c^\infty f(x, a)\,dx$$

converge in the interval $A \leq a \leq B$;

$$\psi(a) = (\varphi)\text{-}\int_c^\infty f(x, a)\, dx$$

(ii) $\partial f/\partial a$ *exist,*

(iii) the function $\partial f/\partial a = f_a(x, a)$

be continuous in the region $c \leq x < \infty,\ A \leq a \leq B$;

(iv) $$\frac{1}{\lambda K}\int_c^\infty \varphi'(x/\lambda)\int_c^x f_a(t, a)\, dt, \quad (1/\lambda > 0)$$

converge uniformly in the interval $A \leq a \leq B$;

(v) $$(\varphi)\text{-}\int_c^\infty f_a(x,a)$$

converge uniformly in the interval $A \leq a \leq B$, *then the function* $\psi(a)$ *has a derivative, the derivative is continuous and is given by the last integral:*

$$\frac{d\psi}{da} = (\varphi)\text{-}\int_c^\infty f_a(x, a)\, dx$$

Proof. Let

$$\chi(a) = (\varphi)\text{-}\int_c^\infty f_a(x, a)\, dx$$

Then, by Theorem 1, $\chi(a)$ is continuous and therefore integrable. By Theorem 2,

$$\int_A^a \chi(a)\, da = (\varphi)\text{-}\int_c^\infty dx \int_A \frac{\partial f}{\partial a} da = (\varphi)\text{-}\int_c^\infty \{f(x, a) - f(x, A)\}\, dx$$

$$= (\varphi)\text{-}\int_c^\infty f(x, a)\, dx - (\varphi)\text{-}\int_c^\infty f(x, A)\, dx.$$

Hence

$$\int_A^a \chi(a)\, da = \psi(a) - \psi(A).$$

From the continuity of $\psi(a)$, it at once follows that

$$\chi(a) = d\psi/da.$$

Two important special cases. *Case I.* Let $\varphi(t) = e^{-t}$ and $c = 0$. As given in example *(ii)* above this is the case of Abel-limit or (A)-limit.

Let $e^{-x/\lambda} f(x, a)$ where λ has any positive value, tend to zero as $x \to \infty$, uniformly throughout the interval $A \leq a \leq B$. The integral

$$\int_0^\infty e^{-x/\lambda} f(x, a) dx$$

is uniformly convergent throughout the region

$$0 < 1/\Lambda \leq 1/\lambda,\ A \leq a \leq B.$$

For then
$$|f| \leq K' e^{\frac{1}{2}x/\lambda}$$
where K' is some constant independent of a and so

$$\left|\int_X^{X'} e^{-x/\lambda} f(x, a)\, dx\right| \leq K' \int_X^{X'} e^{-\frac{1}{2}x/\lambda} dx.$$

Also in this case

$$e^{-x/\lambda} \int_0^x f(t, a) \to 0 \text{ as } x \to 0$$

uniformly in $A \leq a \leq B$ for any positive λ for

$$0 \leq \left| e^{-x/\lambda} \int_0^\infty f(t, a)\, dt \right| \leq e^{-x/\lambda} \int_0^x |f\, dt| \leq K'x\, e^{-\frac{1}{2}x/\lambda}.$$

Under the circumstances

$$\frac{1}{\lambda K} \int_0^\infty dx \varphi'(x/\lambda) \int_0^x f(t, a) dt$$
$$= \frac{1}{\lambda} \int_0^\infty dx e^{-x/\lambda} \int_0^x f(t, a) dt$$
$$= \int_0^\infty e^{-x/\lambda} f(x, a)\, dx$$

Thus the condition (ii) of Theorems 1 and 2 are satisfied and we have the following theorems:

Theorem 1a. *Let*

(i) *$f(x, a)$ be continuous in the region $0 \leq x < \infty$, $A \leq a \leq B$;*

(ii)
$$e^{-x/\lambda} f(x, a) \to 0 \quad as \ x \to \infty$$

uniformly throughout the interval $A \leq a \leq B$, λ being any positive number;

(iii)
$$(A)\text{-}\int_0^\infty f(x, a)\, dx, \text{ i.e., } \lim_{\lambda \to \infty} \int_0^\infty e^{-x/\lambda} f(x, a)\, dx$$

converge uniformly in $A \leq a \leq B$, then the last integral defines a function

$$\psi(a) = (A)\text{-}\int_0^\infty f(x, a)\, dx$$

continuous in the interval $A \leq a \leq B$.

Theorem 2a. *Let*

(i) *$f(x, a)$ be continuous in the region $0 \leq x > \infty$, $A \leq a \leq B$;*

(ii)
$$e^{-x/\lambda} f(x, a) \to 0 \quad as \ x \to \infty$$

uniformly throughout the interval $A \leq a \leq B$,

(iii)
$$(A)\text{-}\int_0^\infty f(x, a)\,dx$$

converge uniformly in the interval $A \leq a \leq B$:

$$\psi(a) = (A)\text{-}\int_0^\infty f(x, a)\,dx$$

then
$$\int_{a_0}^{a_1} \psi(a)\,da = (A)\text{-}\int_0^\infty dx \int_{a_0}^{a_1} f(x, a)\,da$$

or
$$\int_{a_0}^{a_1} da\ (A)\text{-}\int_0^\infty f(x, a)\,dx = (A)\text{-}\int_0^\infty dx \int_{a_0}^{a_1} f(x, a)\,da$$

If $e^{-x/\lambda}\partial f(x, a)/\partial a$ where λ has any positive value, tend to zero as $x \to \infty$, uniformly throughout the interval $A \leq a \leq B$, then reasoning as above we can show that the integral

$$\frac{1}{\lambda K}\int_0^\infty \varphi'(x/\lambda)\int_0^x \frac{\partial f(t, a)}{\partial a}dt$$
$$= \int_0^\infty e^{-x/\lambda} f(x, a)\,dx$$

is uniformly convergent throughout the region

$$0 < 1/\Lambda \leq 1/\lambda, \quad a \leq A \leq B$$

and the condition (iv) of Theorem 3 is satisfied. Thus we have the

Theorem 3a. *Let*

(i)
$$(A)\text{-}\int_0^\infty f(x, a)\,dx,$$

(ii) $\partial f/\partial a$ *exist,*

(iii) the function $\partial f/\partial a = f_a(x, a)$

be continuous in the region $0 \leq x < \infty$, $A \leq a \leq B$

(iv) $e^{-x/\lambda} f_a(x, a) \to 0$ as $x \to \infty$

uniformly throughout the interval $A \leq a \leq B$, λ *being any positive number*

(v)
$$(A)\text{-}\int_0^\infty f_a(x, a)$$

converge uniformly in the interval $A \leq a \leq B$, *then the function* $\psi(a)$ *has a derivative, the derivative is continuous as given by the last integral:*

$$\frac{d\psi}{da} = (A)\text{-}\int_0^\infty f_a(x, a)\,dx. \qquad \text{(Hardy 1912a, Theorem VI, p. 31)}$$

Case II. Let $\varphi(t) = e^{-t^2}$ and $c = 0$. If, in this case $e^{-x^2/\lambda^2} f(x, a)$ where λ has any positive value, tend to zero as $x \to \infty$ uniformly throughout the interval $A \leq a \leq B$, then proceeding in a way similar to that in the Case I we get that the integral

$$\frac{1}{\lambda K} \int_0^\infty \varphi'(x/\lambda) \int_0^x f(t, a)$$

$$= \int_0^\infty e^{-x^2/\lambda^2} f(x, a) dx$$

is uniformly convergent throughout the region

$$0 < 1/\Lambda \leq 1/\lambda, \quad A \leq a \leq B.$$

Thus the analogues of Theorems 1a and 2a hold is the case also.

If $e^{-x^2/\lambda^2} \partial f(x, a)/\partial a$ where λ has any positive value, tend zero as $x \to \infty$, uniformly throughout the interval $A \leq a \leq B$, the reasoning as in the case I we can show that the analogue of Theorem 3a hold also for this case

The convergence factor e^{-x^2/λ^2} has a special advantage as compared to $e^{-x/\lambda}$ that we can study integrals of the type

$$\lim_{\lambda \to \infty} \int_{-\infty}^{+\infty} e^{-x^2/\lambda^2} f(x, a)\, dx$$

directly. This advantage is common to all cases where $\varphi(x)$ is an even function of x.

Generalisation. Let us now generalize the concept of the volume integral:

$$\int f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 \qquad \ldots \qquad (1{\cdot}5)$$

taken over all space. We first integrate over a sphere of radius r, the centre being at a fixed point, so that the integral is now a function of r only, *viz.*

$$\int_{\text{sph}} f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 = F(r)$$

If now (φ)-limit of $F(r)$ as $r \to \infty$ exists, we say that the integral exists and its value is

$$(\varphi)\text{-} \int f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 = (\varphi)\text{-}\lim_{r \to \infty} F(r) \qquad (1{\cdot}6)$$

We may similarly generalize the concept when the integrand contains parameters:

$$f(a_1, a_2, \ldots, a_r, x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 \qquad (1{\cdot}7)$$

taken over all space, as has been done in a previous paper of the author (Ghosh, 1947) In every case there remains some arbitrariness due to the choice of the centre of the sphere of integration. This arbitrariness can be removed by other conditions of the problem.

We may also proceed to study the equation

$$\Delta\psi_q(\mathbf{r}) = = -4\pi \exp[i(\mathbf{qr})] \qquad (1.8)$$

as in one of our previous papers (Ghosh, 1947, p. 25). Reasoning on similar lines and taking (φ)-limit instead of (C, 1)-limit, we arrive at the solution

$$\psi_q(\mathbf{r}) = (\varphi)\text{-}\int \frac{\exp[i\mathbf{qR})]}{|\mathbf{R}-\mathbf{r}|}\, d\Omega_R = \frac{4\pi}{q^2}\exp[i(\mathbf{qr})] \qquad (1{\cdot}9)$$

for a wide class of 'convergence factors' φ's [see below eq. (2·1)].

II. APPLICATIONS

Generalized rules for calculation of matrix-elements. For the purpose of applications to Quantum Mechanics we have to generalize the usual rules for the calculation of matrix-elements of an operator O, *viz.*,

$$O_{mn} = (\varphi)\text{-}\int \psi_m^* O \psi_n d\tau$$

instead of

$$O_{mn} = \int \psi_m^* O \psi_n d\tau$$

taken over the entire configuration space ψ_m and ψ_n are normalized eigen-functions belonging to the m-th and n-th states respectively.

In the solution of problems of Quantum Mechanics often such integrals occur which in the last analysis reduce to

$$\int^\infty k^{2n}\cos ka\,dk, \qquad \int^\infty k^{2n+1}\sin ka\,dk,$$

$$\int^\infty k^{2n+1}\cos ka\,dk, \qquad \int^\infty k^{2n}\sin ka\,dk \quad [cf.\ \text{Dirac (1943)}]$$

The (C, α)-limits of these integrals, for sufficiently large α, are

$$\left.\begin{aligned} &\int^\infty k^{2n}\cos ka\,dk = 0, \qquad \int_0^\infty k^{2n+1}\sin ka\,dk = 0 \\ &\int_0^\infty k^{2n+1}\cos ka\,dk = (-1)^{n+1}(2n+1)!/a^{2n+2} \\ &\int_0^\infty k^{2n}\sin ka\,dk = (-1)^n(2n)!/a^{2n+1} \end{aligned}\right\} \qquad (2{\cdot}1)$$

[*cf.* Ghosh, 1948].

The Abel limits and the limits with $\varphi(x) = e^{-x^2}$ are also the same. But this is no accidental coincidence. By a theorem of Hardy (Hardy, 1912*a*, p. 26, Theorem V) *if* $\varphi(x)$ *be a function of* x *which tends to zero as* $x\rightarrow\infty$ *more rapidly than any power of* x *and has continuous derivatives* $\varphi'(x)$, $\varphi''(x)$, ... $\varphi^{(n+1)}(x)$, *and*

$$\int_0^\infty \varphi^{(n+1)}(x)x^\mu dx, \qquad (n > \mu)$$

is absolutely convergent, then

$$\lim_{\alpha\to 0}\frac{1}{\varphi(0)}\int_0^\infty e^{-imx}x^\mu\varphi(\alpha x)\,dx = \Gamma(\mu+1)m^{-\mu-1}e^{\frac{1}{2}(\mu+1)\pi i} \tag{2·2}$$

provided $m>0$, $\mu>-1$.

The present of series papers by the author shows that under suitable conditions these integrals can be calculated, when μ is a positive integer, by successive differentiation of two integrals only (*cf.*, Ghosh, 1949), *viz.*,

$$\lim_{\alpha\to 0}\frac{1}{\varphi(0)}\int_0^\infty \cos ka\,\varphi(\alpha k)\,dk = 0,\quad \lim_{\alpha\to 0}\frac{1}{\varphi(0)}\int_0^\infty \varphi(\alpha k)\sin ka\,dk = \frac{1}{a},$$

(Hardy, 1912*a*, p. 27) (2·3)

Scattering by a centre of force; Rutherford scattering formula. Our problem is to calculate the scattering of a beam of particles [exp (ikz)] by a field $V(r)$. We wish to obtain an approximate formula. We have to solve the wave equation

$$\Delta\psi+[k^2-U(r)]\psi = 0 \tag{2·4}$$

where $k^2 = 8\pi^2m/h^2$, $U(r) = 8\pi^2mV(r)/h^2$ and where ψ must have the asymptotic form

$$\psi\sim e^{ikz}+r^{-1}e^{ikr}f(\theta). \tag{2·5}$$

Proceeding as in our previous papers (Ghosh, 1947, 1949) and using (φ)-limit, we obtain

$$f(\theta) = -\frac{1}{4\pi}(\varphi)\text{-}\int \exp[ik(\mathbf{n}_0-\mathbf{n},\,\mathbf{r}')]\,U(r')d\tau \tag{2·6}$$

where $\mathbf{n}$ is a unit vector in the direction $\mathbf{r}$, and $\mathbf{n}_0$ is a unit vector in the direction of the positive z-axis so that $z = (\mathbf{n}_0\mathbf{r})$.

For the Coulomb field of the Hydrogen nucleus

$$U(r) = \frac{8\pi^2me}{h^2r}$$

and hence

$$f(\theta) = -\frac{2\pi me}{h^2}(\varphi)\text{-}\int\frac{\exp[ik(\mathbf{n}_0-\mathbf{n},\,\mathbf{r}')]}{r'}\,d\tau \tag{2·7}$$

$$= -\frac{8\pi^2me}{h^2}(\varphi)\text{-}\frac{1}{k^2|\mathbf{n}_0-\mathbf{n}|^2}\quad\text{by (2·1)} \tag{2·8}$$

for a wide class of φ's.

Now $|\mathbf{n}_0-\mathbf{n}| = 2\sin\frac{1}{2}\theta$. Therefore

$$f(\theta) = -\frac{2\pi^2me}{h^2k^2\sin^2\frac{1}{2}\theta}$$

The inntensity scattered into the solid angle $d\omega$ is $|f(\theta)|^2d\omega$ and this gives the Rutherford scattering formula.

Here with the help of (φ)-convergent integrals we have arrived at the result smoothly If some such idea is not brought in, the right hand side of (2·7) oscillates as the integral is taken over an infinite sphere. To overcome this difficulty Wentzel (1927) had to use a modified field

$$V = (c/r) \exp(-r/R)$$

instead of a Coulomb field and then neglect the modificatin under certain physical assumptions. From the mathematical point of view this is equivalent to the introduction of the convergence factor

$$e^{-pr}, \; p = 1/R \geq 0$$

in the integral

$$\int_0^\infty \sin qr\, dr$$

and after integration let $p \to +0$. Thus

$$\int_0^\infty dr e^{-pr} \sin qr = q/(p^2+q^2) \to 1/q$$

as $p \to +0$ and this limit is taken as the value of the integral (Sommerfeld, 1939). This is artificial in as much as a 'convergence factor' is brought in only in the last step to save the situation without enquiring into the effect of its appearance and ultimate disappearance on the previous steps. In our papers referred to above we have developed a theory in which the notion of a convergencefactor is introduced from the very outset and is thus natural. Hardly any special hypothesis of physical nature becomes necessary afterwards. Moreover, the discussion in the present paper shows that follwing Wentzel we could use a modified field

$$V = \frac{e}{r}\varphi(r/R)$$

where φ is any member of class of function satisfying the conditons stated in Hardy's theorem cited above. As this class of functions is sufficiently wide and $(C, 1)$ convergence also leads to same result, practically no physical significance need be attached to such modified fields.

Another example of the application of our theory is furnished by Bethe's investigation regarding the passage of fast particles through matter (Bethe, 1930). There ultimately Bethe had to calculate matrix-elements which involved integrals of the type

$$\int \frac{\exp[i(\mathbf{q}\mathbf{r})]}{|\mathbf{R}-\mathbf{r}|} d\Omega_R$$

over all space and he evaluated it by an unjustifiable interchange of limit processes. In the light of our general discussion of (φ)-convergent integrals, together with boundary conditions and generalized rules for calculation of matrix elements we are at last led to calculate the integral

$$(\varphi)\cdot\int\frac{\exp\,[i(\mathbf{qr})]}{|\mathbf{R}-\mathbf{r}|}\,d\Omega_R$$

subject to certain conditions (*cf.* Ghosh, 1947). As shown above, for sufficiently wide class φ's velue of this integral is

$$\frac{4\pi}{q^2}\exp\,[i(\mathbf{qr})].$$

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SLOW VISCOUS DRAG

By

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Abstract—By a slight modification of the principle of superposition introduced by Seth, the drag suffered by an infinite cylinder whose cross-section is a lemniscate of Bernouilli, moving slowly through a viscous liquid is found to be $8\pi\mu U$

It has been shown by Seth (1942) that when a solid moves slowly through a viscous liquid so that the inertia terms in the equations of motion can be neglected, the solution is obtained by the superposition of two motions, *viz.*

(i) the potential motion of the liquid, regarded as non-viscous, due to the motion of the solid, and

(ii) the motion of the liquid due to a concentrated force applied in the direction of motion of the solid.

In another paper Seth (1954) has shown that all potential motions can be produced by suitable generalized doublets. If $\xi =$ const. represents a family of surfaces in orthogonal curvilinear coordinates, of which the given solid is a member, the solution of the problem in two dimensions, by Seth's method, is of the form

$$\left.\begin{aligned} u &= A\frac{\partial\varphi}{\partial x} - \frac{X}{4\pi\mu}\left[\frac{\partial}{\partial x}(x\psi) - 2\psi\right], \\ v &= A\frac{\partial\varphi}{\partial y} - \frac{X}{4\pi\mu}\frac{\partial}{\partial y}(x\psi), \end{aligned}\right\} \tag{1}$$

where X is the concentrated force, φ is the velocity potential function of the motion (i) and ψ is a function of ξ only satisfying the Laplace's equation $\nabla^2\psi = 0$, and such that the flux across any closed surface is equal to 2π.

The boundary conditions for the viscous liquid motion are

$$u = U, \qquad v = 0 \qquad \text{on the boundary.} \tag{2}$$

These, Seth has shown, can be satisfied if φ is expressible in the form $xf(\xi)$. But, if φ cannot be expressed in this form we must add some more harmonic terms to φ before superposition in the form (1). The boundary conditions (2) can then be satisfied and the concentrated force X determined. This is found to be the drag suffered by the solid. This method is here applied to obtain the drag when an infinite cylinder whose cross-section is a lemniscate of Bernouilli, moves slowly through a viscous liquid.

Consider the transformation

$$\zeta = \tfrac{1}{2}\log\frac{z^2 - c^2}{c^2}, \tag{3}$$

so that
$$(x^2+y^2)^2 = 2c^2(x^2-y^2)+c^4(e^{4\xi}-1), \tag{4}$$
and
$$x^2-y^2-c^2 = 2xy\cot 2\eta. \tag{5}$$

Thus ξ = const. are a confocal family of lemniscates with their foci distant $2c$ apart and η = const. are confocal hyperbolas with the same foci. The lemniscates become a pair of closed curves if $\xi < 0$, while $\xi = 0$ represents the lemniscate of Bernoulli.

The complex potential of the perfect fluid motion is known to be (Basset, 1888)
$$w = B(1+e^{-2\zeta})^{\frac{1}{2}}, \tag{6}$$
so that
$$\varphi = \tfrac{1}{2}B\left[(1+e^{-2(\xi+i\eta)})^{\frac{1}{2}}+(1+e^{-2(\xi-i\eta)})^{\frac{1}{2}}\right]$$
which cannot be expressed in the form $xf(\xi)$.

Since
$$\left.\begin{aligned}\left(x\frac{\partial\xi}{\partial x}\right)_{\xi=0} &= \tfrac{1}{2}(1+\cos\eta+\cos 2\eta+\cos 3\eta)\\ \left(x\frac{\partial\xi}{\partial y}\right)_{\xi=0} &= \tfrac{1}{2}(\sin\eta+\sin 2\eta+\sin 3\eta)\end{aligned}\right\} \tag{7}$$
we add to (6) the terms
$$c\log\{e^{\zeta}+(1+e^{2\zeta})^{\frac{1}{2}}\}+D\log\{e^{-\zeta}+(1+e^{-2\zeta})^{\frac{1}{2}}\}.$$

For a lemniscate $\psi = -\xi$. The equations (1) become
$$\left.\begin{aligned}u &= A_1e^{-\xi}\cos\eta+A_2e^{-2\xi}\cos 2\eta+A_3e^{-3\xi}\cos 3\eta+\frac{X}{4\pi\mu}\left(x\frac{\partial\xi}{\partial x}-\xi\right),\\ v &= A_1e^{-\xi}\sin\eta+A_2e^{-2\xi}\sin 2\eta+A_3e^{-3\xi}\sin 3\eta+\frac{X}{4\pi\mu}\,x\frac{\partial\xi}{\partial y}.\end{aligned}\right\} \tag{8}$$

Substituting (7) in (8) and satisfying the boundary conditions (2) on $\xi = 0$ we get
$$A_1 = A_2 = A_3 = -U;$$
$$X = 8\pi\mu U.$$

The value $8\pi\mu U$ is therefore the drag suffered by the lemniscate.

My thanks are due to Prof. B. R. Seth for his guidance in the preparation of this note.

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THE ELASTIC STRESSES AT THE BOUNDARY OF A SYMMETRICALLY SHAPED HOLE IN AN INFINITE PLATE LOADED BY NORMAL BOUNDARY FORCES IN THE PLANE OF THE PLTAE[1]

By

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Summary. In this paper the elastic stresses at the boundary of a symmetrically shaped hole in an infinite plate, due to bands of force or concentrated forces applied normal to the boundary and in the plane of the plate, are determined. The class of hole shapes considered are those contours in the z-plane into which the unit circle in the ζ-plane is mapped by the function $z = \zeta\left[A_0 + \sum_{n=1}^{n=m} A_n \zeta^{-kn}\right]$ where $k \geq 2$ and A_n's are real coefficients restricted so that the roots of $dz/d\zeta$ lie within the unit circle in the ζ-plane. The force distribution has an angular periodicity of $2\pi/k$. The method of solution consists of a combination of the conformal mapping technique and the Muskhelishvili method (Sokolnikoff, 1941) for solving plane elasticity problems. The results obtained are applicable to finite plates of such an extent that the stresses at the outer boundary due to the loading at the inner boundary, or hole contour, are negligible.

Introduction A problem of considerable importance in stress analysis work is that of determining the elastic stresses in plates loaded at the boundary of a symmetrically shaped hole in the plate by normal bands of force or by normal concentrated forces applied in the plane of the plate. There are several techniques for handling this problem when the hole is circular. The corresponding problem for other boundary shapes is much more difficult to handle. In many instances, however, it is sufficient to know only the stresses at the boundary of hole.

In a recent paper by O. L. Bowie (1950) the stresses at the boundary of a symmetrically shaped hole in an infinite plate due to loading by normal bands of force were determined for a class of hole contours. These results are valid for finite plates of such an extent that the stresses at the outer boundary due to the loading at the hole are negligible. However, in this analysis the bands of normal force are distributed symmetrically with respect to the lines of symmetry of the hole contour. It was expected that by an analagous analysis, the more general problem of the determination of the stresses at the boundary of a symmetrically shaped hole in an infinite plate due to bands of normal force, not necessarily symmetric with respect to the lines of symmetry of the hole contour, could be solved and would include a new and very useful class of problems. Such an analysis has been carried out and constitutes the subject of this paper. The analysis has also been extended to include the case of normal loading by concentrated forces.

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[1] The results presented in this paper were obtained in the course of research conducted at Watertown Arsenal Laboratory, Watertown, Mass,

Basic Assumptions. The mathematical analysis presented here is based on the assumption that the plate is of infinite extent and has zero stresses at infinity.

The hole shapes considered are defined as those contours, τ, in the z-plane into which the unit circle in the ζ-plane is mapped by the function

$$z = \omega(\zeta) = \zeta\left[A_0 + \sum_{n=1}^{n=m} A_n\zeta^{-kn}\right] \tag{1}$$

where $k \geq 2$ and the A_n's are real coefficients restricted so that the roots of $dz/d\zeta$ lie within the unit circle. It follows from the form of this mapping function that the hole contour has an angular periodicity of $2\pi/k$, and is symmetric with respet to the real axis and any line whose direction angle is an integral multiple of π/k. Due to the restrictions imposed on the coefficients, the unit circle and its exterior are mapped conformally onto the hole contour and its exterior.

The distribution of the normal load applied at the hole contour of the plate is assumed to have an angular periodicity of $2\pi/k$. It follows that the resultant force of applied load at the hole vanishes. However, the resultant moment of the applied load does not necessarily vanish. Throughout the analysis the load is assumed to consist of uniform bands of normal force. The corresponding solutions for arbitrary bands of normal force and normal concentrated forces are obtained from the uniform band load solution.

Under the assumptions stated, the problem of normal loading at the hole of the plate may be considered as a plane stress problem, and can be handled by the Muskhelishvili method for solving plane elasticity problems.

Mathematical Analysis. By the introduction of Airy's stress function, plane stress problems in elasticity can be reduced to the solution of the biharmonic equation

$$\nabla^4 U(x, y) = 0$$

subject to the boundary conditions of the problem. As shown by Muskhelishvili (Sokolnikoff, *ibid*, ch. V), the general solution of this equation can be expressed in terms of two analytic functions of the complex variable z; thus

$$U(x, y) = \mathrm{Re}\left\{\bar{z}\Phi(z) + \int^{z} \Psi(z)dz\right\}.$$

The stress functions, $\Phi(z)$ and $\Psi(z)$ can be determined from their analyticity and from the boundary conditions of the problem, in this case, the stress boundary conditions at the hole contour and the stress boundary conditions at ∞.

From the discontinuous character of the loading at the boundary of the hole, certain discontinuities in the stress functions and their derivatives are to be expected. In particular, it will be assumed that $\Phi(z)$ is continuous but that $\Phi'(z)$ and $\Psi(z)$ may have jump discontinuities at this boundary.

From the analyticity of $\Phi(z)$ and $\Psi(z)$ in the region exterior to the hole and from the boundary conditions at ∞, it can be shown (Sokolnikoff, *ibid.* pp. 257-258) that the stress functions are of the form

$$\Phi(z) = \sum_{j=1}^{\infty} a_j z^{-j}$$

$$\Psi(z) = \sum_{j=1}^{\infty} b_j z^{-j}$$

Also, from the symmetry of the stresses, it can be shown that $\Phi'(z) = \Phi'(z e^{2\pi i/k})$

and $$X(z) = \int^{z} \Psi(z)dz = X(ze^{2\pi i/k}),$$ from which it follows that

$$\Phi(z) = \sum_{j=1}^{\infty} \alpha'_j z^{(1-kj)}$$

$$\Psi(z) = \sum_{j=0}^{\infty} \beta'_j z^{-(1+kj)}$$

The α'_j and β'_j are coefficients to be determined from the stress conditions at the hole contour, τ.

Now the boundary condition on τ can be expressed in terms of the stress functions by considering the resultant force acting on a portion of τ originating at z_0 and ending at z. From this consideration it can be shown (Sokolnikoff, *ibid.*, p. 247) that :

$$\int_{z_0}^{z} \sigma_n ds = \left[\Phi(z) + z\overline{\Phi}'(\bar{z}) + \overline{\Psi}(\bar{z})\right]_{z_0}^{z} \quad \text{on } \tau$$

where $\overline{\Psi}(\bar{z})$ is the conjugate of $\Psi(z)$.

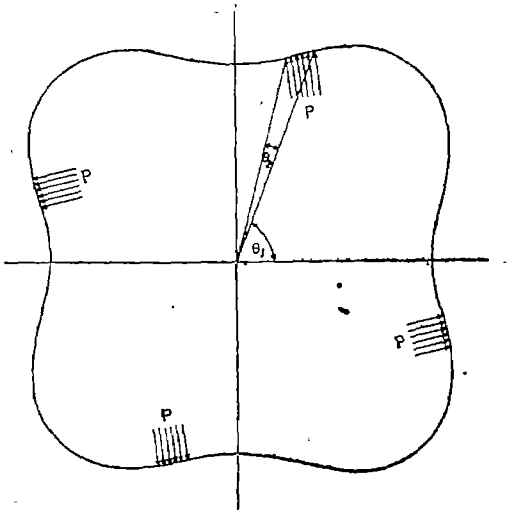

FIG. 1.

Since the loading at the hole consists of periodically spaced bands of uniform force (Fig. 1), $\int_{z_0}^{z'} \sigma_n dz$ is of the form

$$
\begin{aligned}
\int_{z_0}^{z} \sigma_n dz &= C_1 , && -[2\pi/k-\theta_2-\theta_1/2]\leq\theta\leq\theta_1/2,\\
&= -Pz+C_2, && \theta_1/2\leq\theta\leq\theta_1/2+\theta_2,\\
&= C_3 , && \theta_1/2+\theta_2\leq\theta\leq 2\pi/k+\theta_1/2,\\
&= -Pz+C_4, && 2\pi/k+\theta_1/2\leq\theta\leq 2\pi/k+\theta_1/2+\theta_2,\\
&= C_5 , && 2\pi/k+\theta_1/2+\theta_2\leq\theta\leq 4\pi/k+\theta_1/2,\\
&= -Pz+C_6, && 4\pi/k+\theta_1/2\leq\theta\leq 4\pi/k+\theta_1/2+\theta_2,\\
&\cdots \quad \cdots && \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots\\
&= C_{2p-1} , && (p-2)2\pi/k+\theta_1/2+\theta_2\leq\theta\leq(p-1)2\pi/k+\theta_1/2,\\
&= -Pz+C_{2p} , && (p-1)2\pi/k+\theta_1/2\leq\theta\leq(p-1)2\pi/k+\theta_1/2+\theta_2.
\end{aligned}
$$

However, because of the complicated shape of the hole contour, τ, in the z-plane, it is convenient to consider the problem in the ζ-plane for the purpose of applying the stress boundary condition on τ. The hole contour, τ, and its exterior in the z-plane may be mapped into the unit circle and its exterior in the ζ-plane by means of the conformal mapping function of eq. (1). Because of the form of this mapping function, it may be easily verified that

$$\Phi(\omega(\zeta)) = \sum_{j=1}^{\infty} \alpha_j \zeta^{1-kj}$$

$$\Psi(\omega(\zeta)) = \sum_{j=0}^{\infty} \beta_j \zeta^{-(1+kj)} \tag{2}$$

where $\Phi(\omega(\zeta))$ and $\Psi(\omega(\zeta))$ are analytic functions of the complex variable ζ and α_j, β_j are coefficients to be determined from the boundary condition on τ.

In terms of the complex variable, ζ, the boundary condition on τ may be written

$$\Phi(\omega(\sigma)) + \omega(\sigma)\frac{\overline{\Phi'(\omega(\sigma))}}{\overline{\omega'(\sigma)}} + \overline{\Psi(\omega(\sigma))} = f(\varphi) + ig(\varphi) \tag{3}$$

where

$$f(\varphi) + ig(\varphi) \text{ is } \int_{z_0}^{z} \sigma_n dz$$

expressed as a function of the angle φ in the ζ-plane, and $\sigma = e^{i\varphi}$ is an arbitrary point on the unit circle $|\zeta| = 1$. If $\varphi_1/2$ and φ_2 are the angular measures in the ζ-plane corresponding to $\theta_1/2$ and θ_2 in the z-plane,

$$\begin{aligned}
f(\varphi)+ig(\varphi) &= C_1 && -[2\pi/k-\varphi_2-\varphi_1/2]\leq\varphi\leq\varphi_1/2 \\
&= -P\omega(e^{i\varphi})+C_2 && \varphi_1/2\leq\varphi\leq\varphi_1/2+\varphi_2 \\
&= C_3 && \varphi_1/2+\varphi_2\leq\varphi\leq 2\pi/k+\varphi_1/2 \qquad (4) \\
&= -P\omega(e^{i\varphi})+C_4 && 2\pi/k+\varphi_1/2\leq\varphi\leq 2\pi/k+\varphi_1/2+\varphi_2 \\
&= C_5 && 2\pi/k+\varphi_1/2+\varphi_2\leq\varphi\leq 4\pi/k+\varphi_1/2 \\
&= -P\omega(e^{i\varphi})+C_6 && 4\pi/k+\varphi_1/2\leq\varphi\leq 4\pi/k+\varphi_1/2+\varphi_2 \\
&\cdots \qquad \cdots && \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\
&= C_{2p-1} && (p-2)2\pi/k+\varphi_1/2+\varphi_2\leq\varphi\leq(p-1)2\pi/k+\varphi_1/2 \\
&= -P\omega(e^{i\varphi})+C_{2p} && (p-1)2\pi/k+\varphi_1/2\leq\varphi\leq(p-1)2\pi/k+\varphi_1/2+\varphi_2
\end{aligned}$$

where

$$\begin{aligned}
C_1 &= 0 \\
C_2 &= P\omega(e^{i\varphi_1/2}) \\
C_3 &= C_2-P\omega[e^{i(\varphi_1/2+\varphi_2)}] \\
C_4 &= C_3+P\omega[e^{i(2\pi/k+\varphi_1/2)}] \\
C_5 &= C_4-P\omega[e^{i(2\pi/k+\varphi_1/2+\varphi_2)}] \qquad (5) \\
C_6 &= C_5+P\omega[e^{i(4\pi/k+\varphi_1/2)}] \\
&\cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \\
C_{2p-1} &= C_{2p-2}-P\omega[e^{i((p-2)2\pi/k+\varphi_1/2+\varphi_2)}] \\
C_{2p} &= C_{2p-1}+P\omega[e^{i((p-1)2\pi/k+\varphi_1/2)}].
\end{aligned}$$

Alternatively, $f(\varphi)+ig(\varphi)$ may be expressed as complex Fourier series of the form

$$f(\varphi)+ig(\varphi) = \sum_{n=-\infty}^{n=\infty} \varrho_n e^{in\varphi}$$

where

$$\varrho_n = \frac{1}{2\pi}\int_0^{2\pi}[f(\varphi)+ig(\varphi)]e^{-in\varphi}d\varphi.$$

It is found from eqs (4) and (5) that

$$\varrho_n = 0 \quad \text{for} \quad n\neq 0 \quad \text{and} \quad n\neq kq+1 \qquad q=\pm1,\ \pm2,\ldots\ldots$$

$$\varrho_0 = \frac{P}{e^{2\pi i/k}-1}\left\{\omega(e^{i(\frac{1}{2}\varphi_1+\varphi_2)})-\omega(e^{\frac{1}{2}i\varphi_1})\right\}$$

$$\begin{aligned}
\varrho_{1+kq} = {} & \frac{ik}{2\pi(kq+1)}e^{-i(kq+1)\frac{1}{2}\varphi_1}\left\{P\omega(e^{i(\frac{1}{2}\varphi_1+\varphi_2)})e^{-i(kq+1)\varphi_2}-P\omega(e^{\frac{1}{2}i\varphi_1})\right\} \\
& +\frac{iP}{2\pi}\sum_{s=0}^{m}\frac{A_s}{-(q+s)}\left[e^{-ik(q+s)(\frac{1}{2}\varphi_1+\varphi_2)}-e^{-ik(q+s)\frac{1}{2}\varphi_1}\right]
\end{aligned}$$

where $\dfrac{iP}{2\pi}\cdot\dfrac{A_s}{-(q+s)}\left[e^{-ik(q+s)(\frac{1}{2}\varphi_1+\varphi_2)}-e^{-\frac{1}{2}ik(q+s)\varphi_1}\right]$ is taken as $\dfrac{-PA_sk\varphi_2}{2\pi}$ for $q+s=0$.

Moreover, from the forms of the mapping function and the stress functions, eqs. (1) and (2), it is found that the boundary condition on r, eq. (3) may now be written

$$\sum_{i=1}^{\infty} \alpha_j \sigma^{1-kj} + \sum_{j=1}^{\infty} R_j \sigma^{(k(j-m)+1)} + \sum_{j=0}^{\infty} \bar{\beta}_j \sigma^{1+kj} = \sum_{n=-\infty}^{\infty} \varrho_n \sigma^n$$

where the ϱ_n are defined by eq. (5), the R_j's are obtained from

$$R_1 = \gamma_0 \bar{\alpha}_1 (1-k)$$
$$R_2 = \gamma_0 \bar{\alpha}_2 (1-2k) + \gamma_1 \bar{\alpha}_1 (1-k)$$
$$R_3 = \gamma_0 \bar{\alpha}_3 (1-3k) + \gamma_1 \bar{\alpha}_2 (1-2k) + \gamma_2 \bar{\alpha}_1 (1-k)$$

etc.

and the γ's are determined in terms of the mapping coefficients from

$$A_0 \gamma_0 = A_m$$
$$A_0 \gamma_1 + A_1 (1-k) \gamma_0 = A_{m-1}$$
$$A_0 \gamma_2 + A_1 (1-k) \gamma_1 + A_2 (1-2k) \gamma_0 = A_{m-2}$$

etc.

Now, equating coefficients of like powers of σ, yields

$$\left.\begin{array}{ll} \alpha_j + R_{m-j} = \varrho_{1-jk} & (j = 1, 2, \ldots\ldots m-1) \\ \alpha_j = \varrho_{1-jk} & (j \geq m) \end{array}\right\} \tag{7}$$

and

$$\bar{\beta}_j + R_{m+j} = \varrho_{1+jk} \qquad (j = 0, 1, 2, \ldots\ldots). \tag{8}$$

The coefficients, α_j, can be found from eq (7). The first $m-1$ quantities are obtained by the solution of a linear system of simultaneous equations, since the R_j's are linear combinations of the α_j's. Once the α_j's have been determined, the β_j's can be obtained from eq. (8). Since the coefficients α_j, β_j may be determined from eqs. (7) and (8), the stress functions, eq. (2), are now completely determined.

However, in order to determine the boundary stresses, it is sufficiant to know one stress function, $\Phi(\omega(\zeta))$. In particular, it can be shown (Sokolnikoff, *ibid.* p. 247) that the sum of the principal stresses at an arbitrary point, $z = \omega(\zeta)$, of the plate is given by

$$\sigma_r + \sigma_\theta = 4 \,\mathbf{Re}\left\{ \frac{(d/d\zeta)[\Phi(\omega(\zeta))]}{\omega'(\zeta)} \right\}. \tag{9}$$

But, since σ_n is known at the hole contour $(z = \omega(\sigma))$, determination of $\sigma_r + \sigma_\theta$ at this boundary yields a knowledge of both σ_r and σ_θ. $\omega'(\sigma)$ is known from eq. (1), so it remains only to determine $[(d/d\zeta)\{\Phi(\omega(\zeta))\}]$. This is found to be

$$\frac{Pi}{2\pi}\left\{ -\tfrac{1}{2}\log\left|\frac{\sin \frac{1}{4}k(\varphi_1 + 2\varphi)}{\sin \frac{1}{4}k(\varphi_1 - 2\varphi)}\right| + \tfrac{1}{2}\log\left|\frac{\sin \frac{1}{4}k(\varphi_1 + 2\varphi_2 + 2\varphi)}{\sin \frac{1}{4}k(\varphi_1 + 2\varphi_2 - 2\varphi)}\right| + \tfrac{1}{2}\log|2(\cos k\varphi - \cos \tfrac{1}{2}k\varphi_1)| \right.$$

$$\left. - \tfrac{1}{2}\log|2[\cos k\varphi - (\cos \tfrac{1}{2}\varphi_1 + \varphi_2)k]| \right\} \omega'(\sigma) + \frac{kP}{2\pi}\,\omega'(\sigma)(-\varphi_2) + \frac{kP}{2\pi}\,A_0 \varphi_2$$

$$+\frac{P\omega'(\sigma)}{2\pi}\left\{\sum_{n=1}^{\infty}\frac{\sin\frac{1}{2}kn\varphi_1-\sin kn(\frac{1}{2}\varphi_1+\varphi_2)}{n}\cos kn\varphi-\sum_{n=1}^{\infty}\frac{\cos\frac{1}{2}kn\varphi_1-\cos kn(\frac{1}{2}\varphi_1+\varphi_2)}{n}\sin kn\varphi\right\}$$

$$-\sum_{n=1}^{m-1}\sum_{j=1}^{m-n}\frac{PA_{j+n}}{2\pi}(kj-1)\left\{e^{-ikj\varphi}\frac{\sin\frac{1}{2}kn\varphi_1-\sin kn(\frac{1}{2}\varphi_1+\varphi_2)}{n}-ie^{ikj\varphi}\frac{\cos\frac{1}{2}kn\varphi_1-\cos kn(\frac{1}{2}\varphi_1+\varphi_2)}{n}\right\}$$

$$+\frac{kP}{2\pi}\left\{\frac{\sum_{j=0}^{m}A_j[ie^{-ik(j-1)(\frac{1}{2}\varphi_1+\varphi_2)}-ie^{-i(kj(\frac{1}{2}\varphi_1+\varphi_2)+k\varphi)}]}{2[\cos k\varphi-\cos k(\frac{1}{2}\varphi_1+\varphi_2)]}+\frac{\sum_{j=0}^{m}A_j[ie^{-i(kj\frac{1}{2}\varphi_1+k\varphi)}-ie^{-i(\frac{1}{2}k(j-1)\varphi_1)}]}{2[\cos k\varphi-\cos\frac{1}{2}k\varphi_1]}\right\} \tag{10}$$

$$-\frac{kP}{2\pi}\left\{\sum_{j=0}^{m}A_je^{-ikj\varphi}\left[\frac{\sin\frac{1}{2}k\varphi_1}{2[\cos k\varphi-\cos\frac{1}{2}k\varphi_1]}-\frac{\sin k(\frac{1}{2}\varphi_1+\varphi_2)}{2[\cos k\varphi-\cos k(\frac{1}{2}\varphi_1+\varphi_2)]}\right]\right\}+\sum_{j=1}^{m-1}(kj-1)R_{m-j}e^{-ikj\varphi}.$$

where

$$\sum_{n=1}^{\infty}\frac{\sin\frac{1}{2}kn\varphi_1\cos kn\varphi}{n}=\begin{cases}\pi/2-k\varphi_1/4, & -\varphi_1/2<\varphi<\varphi_1/2\\ -k\varphi_1/4, & \varphi_1/2<\varphi<-\varphi_1/2+2\pi/k\end{cases}$$

$$\sum_{n=1}^{\infty}\frac{-\sin kn(\frac{1}{2}\varphi_1+\varphi_2)\cos kn\varphi}{n}=\begin{cases}-\pi/2+\frac{1}{2}k(\varphi_1/2+\varphi_2), & -(\varphi_1/2+\varphi_2)<\varphi<(\varphi_1/2+\varphi_2)\\ \frac{1}{2}k(\varphi_1/2+\varphi_2), & (\varphi_1/2+\varphi_2)<\varphi<-(\varphi_1/2+\varphi_2)+2\pi/k\end{cases}$$

$$\sum_{n=1}^{\infty}\frac{-\cos\frac{1}{2}kn\varphi_1\sin kn\varphi}{n}=\begin{cases}k\varphi/2 & -\varphi_1/2<\varphi<\varphi_1/2\\ -\pi/2+k\varphi/2 & \varphi_1/2<\varphi<-\varphi_1/2+2\pi/k\end{cases}$$

$$\sum_{n=1}^{\infty}\frac{\cos kn(\frac{1}{2}\varphi_1+\varphi_2)\sin kn\varphi}{n}=\begin{cases}-k\varphi/2 & -(\varphi_1/2+\varphi_2)<\varphi<(\varphi_1/2+\varphi_2)\\ \pi/2-k\varphi/2 & (\varphi_1/2+\varphi_2)<\varphi<-(\varphi_1/2+\varphi_2)+2\pi/k.\end{cases}$$

Thus $\sigma_r+\sigma_\theta$ at the boundary, due to loading by uniform bands of normal force, is completely determined by eqs. (1), (9) and (10). This result agrees with that obtained by O. L. Bowie (*l.c.*) for the case of uniform bands of normal force distributed symmetrically with respect to the lines of symmetry of the hole contour. The boundary stresses due to an arbitrary distribution of normal force can be solved from the results presented here, i.e., by superposition of appropriate band load solutions.

To obtain the corresponding solution for the case of normal concentrated forces of magnitude F, applied at $\theta=\theta_1/2$, $\theta_1/2+2\pi/k$, $\theta_1/2+4\pi/k$ etc , P is replaced by

$$P=\frac{F}{\int_{\frac{1}{2}\varphi_1}^{\frac{1}{2}\varphi_1+\varphi_2}|\omega'(e^{i\varphi})|d\varphi}$$

and the limit of the above solution is taken as $\varphi_2\to0$. As a result we obtain for the sum of the principal stresses

$$\sigma_r+\sigma_\theta=\frac{2kF}{\pi|\omega'(e^{i\varphi_1/2})|}\mathrm{Re}\left\{-\tfrac{1}{2}+\frac{A_0}{\omega'(\sigma)}+\sum_{n=1}^{m-1}\sum_{j=1}^{m-n}\frac{A_{j+n}(kj-1)}{\omega'(\sigma)}[e^{-ikj\varphi}\cos\tfrac{1}{2}kn\varphi_1+ie^{ikj\varphi}\sin\tfrac{1}{2}kn\varphi_1]\right.$$

$$+\frac{2\pi|\omega'(e^{\frac{1}{2}i\varphi_1})|}{kF}\sum_{j=1}^{m-1}\frac{(kj-1)}{\omega'(\sigma)}R_{m-j}e^{-ikj\varphi}+\frac{k}{2}\sum_{j=0}^{m}\Big[-A_je^{-ikj\varphi}(1-\cos k\varphi\cos\tfrac{1}{2}k\varphi_1)+A_je^{-\frac{1}{2}ik(j-1)\varphi_1}\times$$

$$\frac{\times\{(j-1)[\cos k\varphi-\cos\frac{1}{2}k\varphi_1]-i\sin\frac{1}{2}k\varphi_1\}+A_je^{-i(\frac{1}{2}kj\varphi_1+k\varphi)}\{-j[\cos k\varphi-\cos\frac{1}{2}k\varphi_1]+i\sin\frac{1}{2}k\varphi_1\}}{[\cos k\varphi-\cos\frac{1}{2}k\varphi_1]^2\omega'(\sigma)}\Big]\Big\}$$

In the particular case of a hole in the shape of a hypotrochoid

$$z=\omega(\sigma)=A_0e^{i\varphi}+A_1e^{i(1-k)\varphi}.$$

For the case of loading by uniform bands of force

$$\sigma_r+\sigma_\theta=4\mathbf{Re}\Big\{\frac{-kP}{2\pi}\varphi_2+\frac{kP}{2\pi}\frac{A_0\varphi_2}{\omega'(\sigma)}$$

$$+\frac{P}{2\pi}\Big[\sum_{n=1}^{\infty}\frac{\sin\frac{1}{2}kn\varphi_1-\sin kn(\frac{1}{2}\varphi_1+\varphi_2)}{n}\cos kn\varphi-\sum_{n=1}^{\infty}\frac{\cos\frac{1}{2}kn\varphi_1-\cos kn(\frac{1}{2}\varphi_1+\varphi_2)}{n}\sin kn\varphi\Big\}\Big]$$

$$-4\mathbf{Re}\Big\{\frac{kP}{2\pi}\frac{[\cos k(\frac{1}{2}\varphi_1+\varphi_2)-\cos\frac{1}{2}k\varphi_1]\sin k\varphi}{2[\cos k\varphi-\cos k(\frac{1}{2}\varphi_1+\varphi_2)][\cos k\phi-\cos\frac{1}{2}k\varphi_1]}\Big[\frac{A_0+A_1e^{-ik\varphi}}{\omega'(\sigma)}\Big]\Big\}.$$

For the case of concentrated loads

$$\sigma_r+\sigma_\theta=\mathbf{Re}\frac{kF}{\pi|\omega'(e^{\frac{1}{2}i\varphi_1})|}\Big\{-1+\frac{2A_0}{\omega'(\sigma)}+\frac{k\sin\frac{1}{2}k\varphi_1\sin k\varphi[A_0+A_1e^{-ik\varphi}]}{\omega'(\sigma)[\cos k\varphi-\cos\frac{1}{2}k\varphi_1]^2}\Big\}.$$

Conclusion. It was expected that, by a combination of the conformal mapping technique and the Muskhelishvili method for solving plane elasticity problems, it would be possible to derive compact expressions for the stresses at the boundry of a symmetrically shaped hole in an infinite plate loaded at the boundary by normal bands of force or by normal concentrated forces applied in the plane of the plate. Previous work by O. L. Bowie has demonstrated that this is true in the special case of bands of normal force distributed symmetrically with respect to the lines of symmetry of the hole contour. It was also expected that the results presented in this paper would be applicable to many practical problems involving finite plates. In partiular, the results are applicable to finite plates of such a size that the stresses at the outer boundary, due to loading at the boundary of the hole, are negligible.

The boundary stresses were derived for a wide and practical class of boundary contours. It was found that these stresses assume a very simple form in the case of a hole in the shape of a hypotrochoid loaded by periodically spaced concentrated forces. The results obtained in this paper agree with those obtained by O. L Bowie for the particular case of loading by bands of normal force distributed symmetrically with respect to the lines of symmetry of the hole.

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ON SOME RESULTS INVOLVING GENERALISED LAPLACE'S TRANSFORMS

By

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The object of this paper is to obtain a result involving a Hypergeometric function and Generalised k-function of Bateman, and some theorems which are generalisations of results given by Humbert (1937) and Dineschhandra (1952). The results are believed to be new.

1. The Laplace integral

$$\varphi(p) = p\int_0^\infty e^{-px} f(x)\,dx \tag{1.1}$$

has been generalised by Meijer (1941) in the form

$$\varphi(p) = p\int_0^\infty e^{-\frac{1}{2}px}(px)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},\, m}(px) f(x)\,dx \qquad (1,2)$$

which, when $k = m$ reduces to (1.1).

Jaiswal (1952) has symbolically denoted the integral (1.2) as

$$f(x) \xrightarrow[m]{k+\frac{1}{2}} \varphi(p). \tag{1.3}$$

Varma (1947, 1951) has also given two generalisations of the integral (1.1).

They are

$$\varphi(p) = p\int_0^\infty (2px)^{-\frac{1}{4}} W_{k,\, m}(2px) f(x)\,dx \tag{1.4}$$

and

$$\varphi(p) = p\int_0^\infty e^{-\frac{1}{2}px}(px)^{m-\frac{1}{2}} W_{k,\, m}(px) f(x)\,dx \tag{1.5}$$

The integral (1.4) has been symbolically denoted by Bose (1949) as

$$\varphi(p) \overset{k}{\underset{m}{=}} f(x). \tag{1.6}$$

When the substitutions $k = \frac{1}{4}$ and $m = \pm\frac{1}{4}$ are made in (1.4) and $k = -m + \frac{1}{2}$ in (1.5), they reduce to (1.1).

In each of the above integral equations $\varphi(p)$ is said to be the image of $f(x)$ and $f(x)$ the original of $\varphi(p)$, the transform in (1.2) is called Meijer Transform and the transform in (1.4) is called Whittaker Transform.

2. We have due to Chakrabarti (1952, p. 67) an operational representation

$$f(p) = \Gamma(l+n+1)\Gamma(l-n+1)p^{-2l}e^{\frac{1}{2}p}K_{2n}^{2l}(\tfrac{1}{2}p)$$

$$\doteqdot x^{l-n}(1+x)^{l+n} = h(x)$$

and on term by term interpretation we obtain

$$p^{-2m}h\left(\frac{1}{p}\right) = \frac{1}{p^{2m-n+l}}\,{}_1F_0\left[-(l+n);\ -\frac{1}{p}\right]$$

$$\doteqdot \frac{x^{2m-n+l}}{\Gamma(2m-n+l+1)}\,{}_1F_1\left[\begin{matrix}-(l+n)\\(2m-n+l+1)\end{matrix};\ -x\right]$$

$$= g(x),\quad R(p) > 0,\quad R(2m-n+l+1) > 0.$$

And further

$$g(x)x^{\frac{1}{2}-m-k} = x^{m-n+l-k+\frac{1}{2}}\,{}_1F_1\left[\begin{matrix}-(l+n)\\(2m-n+l+1)\end{matrix};\ -x\right]$$

$$\doteqdot \frac{\Gamma(m-n+l-k+3/2)}{p^{m-n+l-k+\frac{1}{2}}}\,{}_2F_1\left[\begin{matrix}-(l+n),\ (m-n+l-k+3/2)\\(2m-n+l+1)\end{matrix};\ -\frac{1}{p}\right]$$

$$= F(p),\quad R(p) > 0,\ R(m-n+l-k+3/2) > 0.$$

Hence due to Harishanker's theorem (1948) we obtain

$$\Gamma(l+n+1)\Gamma(l-n+1)p^{-2l}e^{\frac{1}{2}p}K_{2n}^{2l}(\tfrac{1}{2}p)$$

$$= p^{m-k+\frac{1}{2}}\Gamma(m-n+l-k+3/2)\int_0^\infty e^{-\frac{1}{2}u}u^{k-1}W_{k,m}(u)$$

$$\times\left(\frac{u}{p}\right)^{m-n+l-k+\frac{1}{2}}\,{}_2F_1\left[\begin{matrix}-(l+n),\ (m-n+l-k+3/2)\\(2m-n+l+1)\end{matrix};\ -\frac{u}{p}\right]du$$

and therefore

$$\Gamma(l+n+1)\Gamma(l-n+1)p^{-2l}e^{\frac{1}{2}p}K_{2n}^{2l}(\tfrac{1}{2}p)/\Gamma(m-n+l-k+3/2)$$

$$= p\int_0^\infty e^{-\frac{1}{2}pt}(pt)^{m-\frac{1}{2}}t^{n-l}\,{}_2F_1\left[\begin{matrix}-(l+n),\ m-n+l-k+3/2)\\(2m-n+l+1)\end{matrix};\ -t\right]\times W_{k,m}(pt)dt, \quad (2.1)$$

$R(p) > 0$, $R(2m-n+l+1) > 0$, $R(m-n+l-k+3/2) > 0$, $(l-n+1) > 0$ and $l \geq -1$.

This is an illustration of (1 5).

3. Theorem I. *If*

$$f(t)\xrightarrow[m]{k+\frac{1}{2}}\varphi(s)$$

and

$$f(s) \doteqdot g(t),$$

then

$$\varphi(s) = \int_0^\infty g(u)\, \chi(u,s)du \tag{3.1}$$

*where**

$$\chi(u, s) = \frac{1}{s} \frac{\Gamma_*(2-k\pm m)}{\Gamma(2-2k)}\, {}_2F_1\left[\begin{matrix} 2-k\pm m \\ 2-2k \end{matrix};\ -\frac{u}{s}\right] \tag{3.2}$$

provided

(i) $R(2-k\pm m) > 0, \quad R(s) > s_0 > 0;$

(ii) $R(\mu_1+1) > 0,$ *where* $g(u) = O(u^{\mu_1})$ *for small* u;

(iii) $g(u)\chi(u, s)$ *tends to zero when* $u \to \infty$ *for* $R(s) > s_0 \geq 0$; *and*

(iv) *the Laplace transform of* $|g(t)|$ *exists.*

Proof. We have

$$\varphi(s) = s \int_0^\infty e^{-\frac{1}{2}st}(st)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},\, m}(st)f(t)dt$$

and

$$f(t) = t \int_0^\infty e^{-tu}g(u)du$$

Hence,

$$\varphi(s) = s \int_0^\infty e^{-\frac{1}{2}st}(st)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},\, m}(st)\left[t \int_0^\infty e^{-tu}g(u)du\right]dt$$

$$= s^{-k+\frac{1}{2}} \int_0^\infty g(u)\left[\int_0^\infty e^{-(u/s+\frac{1}{2})st}t^{-k+\frac{1}{2}} W_{k+\frac{1}{2},\, m}(st)dt\right]du$$

Now on integrating we get, from [Goldstein, 1932, p. 112]

$$\varphi(s) = \frac{1}{s} \frac{\Gamma_*(2-k\pm m)}{\Gamma(2-2k)} \int_0^\infty g(u)\, {}_2F_1\left[\begin{matrix} 2-k\pm m \\ 2-2k \end{matrix};\ -\frac{u}{s}\right]du$$

$$= \int_0^\infty g(u)\chi(u, s)du.$$

Regarding the change in the order of integration in (3.3) we observe that the t integral is absolutely convergent if $R(2-k\pm m) > 0$ and $R(s) > s_0 > 0$; the u integral is absolutely convergent if the Laplace transform of $|g(u)|$ exists; and the repeated integral

$$\int_0^\infty g(u)\chi(u, s)du$$

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* $\Gamma(a\pm b)$ denotes $\Gamma(a+b)\,\Gamma(a-b)$; and

$${}_2F_1\left[\begin{matrix} a\pm b \\ c \end{matrix};\ x\right] \text{ denotes } {}_2F_1\left[\begin{matrix} a+b,\ a-b \\ c \end{matrix};\ x\right]$$

is absolutely convergent if $R(\mu_1+1)>0$ where $g(u)\chi(u,s)$ tends to zero as u tends to ∞ for $R(s)>s_0>0$. Hence the change in the order of integration is justified by the de Vallée Poussin's theorem.

4. If in the above theorem we put $k=\pm m$ we get the following known result in Laplace transform (Humbert, 1937)

If $$\varphi(p) \doteqdot f(x)$$

and $$f(p) \doteqdot g(x)$$

then $$\frac{1}{p}\varphi(p) = \int_0^\infty \frac{g(t)}{(p+t)^2}dt. \tag{4.1}$$

5. *Example.* Let

$$g(t) = t^n \doteqdot \frac{\Gamma(n+1)}{s^n} = f(s), \quad R(n+1)>0,$$

then with the help of Goldstein's integral, we obtain

$$\varphi(s) = \frac{\Gamma(n+1)\Gamma_*(1-n-k\pm m)}{\Gamma(1-n-2k)}s^n,$$

$$R(1-n-k\pm m)>0.$$

And therefore from theorem I,

$$\int_0^\infty u^n\,{}_2F_1\left[\begin{matrix}(2-k\pm m)\\(2-2k)\end{matrix};-\frac{u}{s}\right]du$$

$$= \frac{\Gamma(n+1)\Gamma_*(1-n-k\pm m)\Gamma(2-2k)}{\Gamma(2-k\pm m)\Gamma(1-n-2k)}s^{n+1} \tag{5.1}$$

$R(s)>0$, $R(n+1)>0$, $R(1-n-k\pm m)>0$ and $R(2-k\pm m)>0$.

6. Theorem II, *If* $$x^{\nu-1}f_1(x)\xrightarrow[m_1]{k_1+\frac{1}{2}}\varphi_1(p)$$

$$f_1(p)\overset{k}{\underset{m}{=}}f_2(x)$$

and $$p^{1-\lambda+m}f_2(p^\mu) \doteqdot g(x), \quad \mu \geqq 1.$$

then $$\varphi_1(p) = \int_0^\infty g(s)\theta(s,p)ds$$

where $$x^{\nu}W_\lambda^\mu\left(\frac{x}{s^\mu}\right)\xrightarrow[m_1]{\lambda_1+\frac{1}{2}}\theta(s,p)$$

and $$W_\lambda^\mu\left(\frac{x}{s^\mu}\right) = \frac{1}{(2x)^{\lambda/\mu}}\sum_{s=0}^\infty\frac{(-)^r\Gamma_*(\lambda/\mu+r/\mu+\frac{1}{4}\pm m)}{r!\,\Gamma(\lambda/\mu+r/\mu+3/4-k)}$$

$$\times\left(\frac{s^\mu}{2x}\right)^{r/\mu}{}_2F_1\left[\begin{matrix}\lambda/\mu+r/\mu+\frac{1}{4}\pm m\\\lambda/\mu+r/\mu+3/4-k\end{matrix};\frac{1}{2}\right]$$

provided $$R\left(\frac{\lambda}{\mu}+\tfrac{1}{4}\pm m\right)>0,\quad R(\nu\pm m\pm m_1-k_1+5/4)>0$$

and $$R(\mu_1+1)>0 \quad \textit{where } g(s)=O(s^{\mu_1}),$$

$$g(s)W_\lambda^\mu\left(\frac{t_0}{s^\mu}\right) \textit{ is bounded for } t>t_0>0,$$

$R(p)>p_0>0$, *and the integral in* (6.1) *is absolutely convergent.*

Proof. We have [Bose 1949, p. 56], if

$$f_1(p)\overset{k}{\underset{m}{=}}f_2(x)$$

and $$p^{1-\lambda+\mu}f_2(p^\mu) \doteqdot g(x)$$

then, provided the integral is convergent,

$$f_1(p)=p\int_0^\infty W_\lambda^\mu\left(\frac{p}{s^\mu}\right)g(s)ds, \quad \mu\geq 1.$$

Also $$\varphi_1(p)=p\int_0^\infty e^{-\frac{1}{2}pt}(pt)^{-k_1-\frac{1}{4}}W_{k_1+\frac{1}{4},\, m_1}(pt)\times t^{\nu-1}f_1(t)\,dt.$$

Hence $$\varphi_1(p)=p\int_0^\infty e^{-\frac{1}{2}pt}(pt)^{-k_1-\frac{1}{4}}W_{k_1+\frac{1}{4},\, m_1}(pt)t^\nu\left[\int_0^\infty W_\lambda^\mu\left(\frac{t}{s^\mu}\right)g(s)ds\right]dt.$$

$$=\int_0^\infty g(s)\left[p\int_0^\infty e^{-\frac{1}{2}pt}(pt)^{-k_1-\frac{1}{4}}W_{k_1+\frac{1}{4},\, m_1}(pt)t^\nu W^\mu\left(\frac{t}{s^\mu}\right)dt\right]ds$$

$$=\int_0^\infty g(s)\theta(s,p)ds$$

where $$t^\nu W_\lambda^\mu\left(\frac{t}{s^\mu}\right)\xrightarrow[m_1]{k_1+\frac{1}{4}}\theta(s,p)$$

In order to justify the change of the order of integration, we observe that the s-integral is absolutely convergent, if $R(\mu_1+1)>0$ where $g(s)=O(s^{\mu_1})$ and $g(s)W_\lambda^\mu\left(\frac{t_0}{s^\mu}\right)$ is bounded for $t>t_0>0$. The t-integral is absolutely convergent, if $R(\nu\pm m\pm m_1-k_1+5/4)>0$ and $R(p>p_0>0$, the repeated integral exists due to the convergence of (6. 1)

Thus, we see that by de la Vallée Poussin's theorem the change of order of integration is justified,

7. **Corollary.** *If* $k_1=m_1$, *then we obtain the theorem due to Dineshchandra (1952)*

8. **Example.** Let $\mu=1$ and $g(x)=x^n$, then

$$W_\lambda^1\left(\frac{x}{s}\right)=\frac{1}{(2x)^\lambda}\frac{\Gamma_*(\lambda+\frac{1}{4}\pm m)}{\Gamma(\lambda-k+3/4)}\;{}_2F_1\left[\begin{matrix}\lambda+\frac{1}{4}\pm m\\ \lambda+3/4-k\end{matrix};\tfrac{1}{2}-\frac{s}{2x}\right]$$

and
$$g(x) = x^n \doteqdot \frac{\Gamma(n+1)}{p^n} = p^{2-\lambda} f_2(p),$$

$$R(p) \geq 0, \quad R(n+1) > 0,$$

therefore [Bose 1949, p. 12]
$$f_2(x) = \Gamma(n+1) x^{\lambda-n-2}$$

$$\overset{k}{\underset{m}{-}} \frac{\Gamma(n+1)}{p^{\lambda-n-2}} \frac{\Gamma_*(\lambda-n-3/4\pm m)}{2^{\lambda-n-1}\Gamma(\lambda-n-k-\frac{1}{4})} {}_2F_1\left[\begin{matrix}\lambda-n-3/4\pm m \\ \lambda-n-k-\frac{1}{4}\end{matrix} ; \frac{1}{2}\right] = f_1(p),$$

$$R(p) > 0, \quad R(\lambda-n-3/4\pm m) > 0;$$

and
$$\varphi_1(p) = \frac{\Gamma(n+1)\Gamma_*(\lambda-n-\frac{3}{4}\pm m)}{2^{\lambda-n-1}\Gamma(\lambda-n-k-\frac{1}{4})} {}_2F_1\left[\begin{matrix}\lambda-n-3/4\pm m \\ \lambda-n-k-\frac{1}{4}\end{matrix} ; \frac{1}{2}\right]$$
$$\times \frac{\Gamma_*(\nu-\lambda+n-2k_1+\frac{3}{2}\pm m)}{\Gamma(\nu-\lambda+n-2k_1+2)} \frac{1}{p^{\nu-\lambda+n+1}}$$

$$\xrightarrow[m_1]{k_1+\frac{1}{4}} \frac{\Gamma(n+1)\Gamma_*(\lambda-n-\frac{3}{4}\pm m)}{2^{\lambda-n-1}\Gamma(\lambda-n-k-\frac{1}{4})} x^{\nu-\lambda+n+1} \times {}_2F_1\left[\begin{matrix}\lambda-n-3/4\pm m \\ \lambda-n-k-\frac{1}{4}\end{matrix} ; \frac{1}{2}\right] = x^{\nu-1} f_1(x)$$

$$R(p) > 0, \quad R(\nu-\lambda+n-2k_1+3/2\pm m_1) > 0.$$

Hence from (6.1) we get
$$\int_0^\infty \int_0^\infty s^n e^{-\frac{1}{2}pt}(pt)^{-k_1-\frac{1}{4}} W_{k_1+\frac{1}{4}, m_1}(pt) t^{\nu-\lambda} {}_2F_1\left[\begin{matrix}\lambda+\frac{1}{4}\pm m \\ \lambda-k+\frac{3}{4}\end{matrix} ; \frac{1}{2}-\frac{s}{2t}\right] ds\, dt$$
$$= \frac{2^{n+1}\Gamma(n+1)\Gamma_*(\lambda-n\pm m-3/4)\Gamma_*(\nu-\lambda+n-2k_1+3/2\pm m_1)}{\Gamma(\lambda-n-k-\frac{1}{4})\Gamma(\nu-\lambda+n-2k_1+2)}$$
$$\times \frac{\Gamma(\lambda-k+3/4)}{\Gamma_*(\lambda\pm m+\frac{1}{4})} {}_2F_1\left[\begin{matrix}\lambda-n-3/4\pm m \\ \lambda-n-k-\frac{1}{4}\end{matrix} ; \frac{1}{2}\right] \frac{1}{p^{\nu+n-\lambda+1}}$$
$$R(\lambda+\tfrac{1}{4}\pm m) > R(n+1) > 0, \quad R(p) \geq 0$$
$$R(\nu\pm m\pm m_1-k_1+5/4) > 0 \quad \text{and} \quad R(\nu-\lambda+n-2k_1+3/2\pm m_1) > 0.$$

9. Theorem III. *If*
$$\varphi_1(p) = p\int_0^\infty e^{-\frac{1}{2}pt}(pt)^{m_1-\frac{1}{2}} W_{k_1, m_1}(pt) t^{\nu-1} f_1(t)\, dt$$

$$f_1(p) \overset{k}{\underset{m}{-}} f_2(x)$$

and
$$p^{1-\lambda+\mu} f_2(p^\mu) \doteqdot g(x), \quad \mu \geq 1,$$

then
$$\varphi_1(p) = \int_0^\infty g(s) T(s, p)\, ds$$

where
$$T(s,\, p) = p\int_0^\infty e^{-\frac{1}{2}pt}(pt)^{m_1-\frac{1}{2}}W_{k_1,\, m_1}(pt)t^{\nu}W_\lambda^\mu\left(\frac{t}{s^\mu}\right)dt \tag{9.1}$$

and
$$W_\lambda^\mu\left(\frac{x}{s^\mu}\right) = \frac{1}{(2x)^{\lambda/\mu}}\sum_{r=0}^{\infty}\frac{(-)^r\Gamma_*(\lambda/\mu+r/\mu+\frac{1}{4}\pm m)}{r\,!\,\Gamma(\lambda/\mu+r/\mu+\frac{3}{4}-k)}\times\left(\frac{s^\mu}{2x}\right)^{r/\mu}$$
$$\times\, {}_2F_1\left[\begin{matrix}\lambda/\mu+r/\mu+\frac{1}{4}\pm m\\ \lambda/\mu+r/\mu+\frac{3}{4}-k\end{matrix}\,;\,\frac{1}{2}\right],$$

provided $R(\lambda/\mu+\frac{1}{4}\pm m)>0$, $R(\nu+m_1\pm m_1\pm m+5/4)>0$ *and* $R(\mu_1+1)>0$ *where* $g(s) = O(s^{\mu_1})$, $g(s)W_\lambda^\mu\left(\frac{t_0}{s^\mu}\right)$ *is bounded for* $t>t_0>0$, $R(p)>p_0>0$, *and the integral in* (9.1) *is absolutely convergent.*

The proof is similar as in Theorem II.

10. Corollary. *If in Theorem III, we put* $k_1 = -m_1+\frac{1}{2}$, *then we get the theorem due to Dineshchandra* (1952).

11. *Example.* Let $\mu = 1$ and $g(x) = x^n$, then
$$\varphi_1(p) = \frac{\Gamma(n+1)\Gamma_*(\lambda-n-\frac{3}{4}\pm m)\Gamma_*(\nu-\lambda+n+m_1\pm m_1+2)}{2^{\lambda-n-1}\Gamma(\lambda-n-k-\frac{1}{4})\Gamma(\nu-\lambda+n+m_1-k_1+5/2)}\times$$
$$\frac{1}{p^{\nu-\lambda+n+1}}\,{}_2F_1\left[\begin{matrix}\lambda-n-\frac{3}{4}\pm m\\ \lambda-n-k-\frac{1}{4}\end{matrix}\,;\,\frac{1}{2}\right]$$

Hence from (9.1) we get
$$\int_0^\infty\int_0 s^n e^{-\frac{1}{2}pt}(pt)^{m_1-\frac{1}{2}}W_{k_1,\, m_1}(pt)t^{\nu-\lambda}\,{}_2F_1\left[\begin{matrix}\lambda+\frac{1}{4}\pm m\\ \lambda-k+\frac{3}{4}\end{matrix}\,;\,\frac{1}{2}-\frac{s}{2t}\right]ds\,dt$$
$$= \frac{2^{n+1}\Gamma(n+1)\Gamma(\lambda-k+\frac{3}{4})\Gamma_*(\lambda-n-\frac{3}{4}\pm m)\Gamma_*(\nu-\lambda+n+m_1\pm m_1+2)}{\Gamma_*(\lambda+\frac{1}{4}\pm m)\Gamma(\lambda-n-k-\frac{1}{4})\Gamma(\nu-\lambda+n+m_1-k+5/2)}$$
$$\times\frac{1}{p^{\nu-\lambda+n+2}}\,{}_2F_1\left[\begin{matrix}\lambda-n-\frac{3}{4}\pm m\\ \lambda-n-\frac{1}{4}-k\end{matrix}\,;\,\frac{1}{2}\right]$$
$$R(p)>0,\quad R(\lambda+\tfrac{1}{4}\pm m)>R(n+1)>0,$$
$$R(\nu+m_1\pm m_1\pm m+5/4)>0 \text{ and } R(\nu-\lambda+n+m_1\pm m_1+2)>0.$$

12.. Theorem IV. *If*
$$\varphi_1(p)\,\underset{m_1}{\overset{k_1}{=}}\,x^{\nu-1}f_1(x)$$
$$f_1(p)\underset{m}{\overset{k}{=}}f_2(x)\quad \mu\geq 1$$

and
$$p^{1-\lambda+\mu}f_2(p^\mu)\doteqdot g(x),$$

then
$$\varphi_1(p) = \int_0^\infty g(s)Z(s,\, p)ds$$

where
$$Z(s,\, p) \overset{k_1}{\underset{m_1}{=}} x^{\nu} W_{\lambda}^{\mu}\left(\frac{x}{s^{\mu}}\right)$$

and
$$W_{\lambda}^{\mu}\left(\frac{x}{s^{\mu}}\right) = \frac{1}{(2x)^{\lambda/\mu}} \sum_{r=0}^{\infty} \frac{(-)^r \Gamma_{*}(\lambda/\mu + r/\mu + \frac{1}{4} \pm m)}{r\,!\,\Gamma(\lambda/\mu + r/\mu + \frac{3}{4} - k)} \left(\frac{s^{\mu}}{2x}\right)^{r/\mu} {}_2F_1\left[\begin{matrix} \lambda/\mu + r/\mu + \frac{1}{4} \pm m \\ \lambda/\mu + r/\mu + \frac{3}{4} - k \end{matrix} ;\, \tfrac{1}{2}\right]$$

provided $R(\lambda/\mu + \frac{1}{4} \pm m) > 0, \quad R(\nu \pm m \pm m_1 + 3/2) > 0,$

and $R(\mu_1 + 1) > 0,$

where $g(s) = O(s^{\mu_1}), \quad g(s) W_{\lambda}^{\mu}\left(\frac{t_0}{s^{\mu}}\right)$

is bounded for $t > t_0 > 0$, $R(p) > p_0 > 0$, *and the integral in* (12.1) *is absolutely convergent.*

The proof is similar as in Theorem II.

13. Corollary. *When in the above theorem we put* $k_1 = \frac{1}{4}$ *and* $m_1 = \pm\frac{1}{4}$, *then we get the result due to Dineshchandra* (1952).

14. Example. Let $\mu = 1$ and $g(x) = x^{n}$, then

$$\varphi_1(p) = \frac{1}{2^{\nu+1}} \frac{\Gamma(n+1)\Gamma_{*}(\lambda - n - \frac{3}{4} \pm m)\Gamma_{*}(\nu - \lambda + n \pm m_1 + 9/4)}{\Gamma(\lambda - n - k - \frac{1}{4})\Gamma(\nu - \lambda + n - k_1 + 11/4)}$$

$${}_2F_1\left[\begin{matrix} \lambda - n - \frac{3}{4} \pm m \\ \lambda - n - k - \frac{1}{4} \end{matrix} ;\, \tfrac{1}{2}\right] {}_2F_1\left[\begin{matrix} \nu - \lambda + n \pm m_1 + 9/4 \\ \nu - \lambda + n - k_1 + 11/4 \end{matrix} ;\, \tfrac{1}{2}\right]$$

$$R(p) > 0, \quad R(\nu - \lambda + n \pm m_1 + 9/4) > 0.$$

Hence from (12.1) we get

$$\int_0^{\infty}\int_0^{\infty} s^{n}(2pt)^{-\frac{1}{4}} W_{k_1,\, m_1}(2pt)\, t^{\nu-\lambda}\, {}_2F_1\left[\begin{matrix} \lambda + \frac{1}{4} \pm m \\ \lambda - k + 3/2 \end{matrix} ;\, \tfrac{1}{2} - \frac{s}{2t}\right] ds\, dt$$

$$= \frac{1}{2^{\nu-\lambda+1}} \frac{\Gamma(n+1)\Gamma(\lambda - k + \frac{3}{4})\Gamma_{*}(\lambda - n - \frac{3}{4} \pm m)\Gamma_{*}(\nu - \lambda + n \pm m_1 + \frac{9}{4})}{\Gamma_{*}(\lambda + \frac{1}{4} \pm m)\Gamma(\lambda - n - k - \frac{1}{4})\Gamma(\nu - \lambda + n - k_1 + 11/4)}$$

$$\times {}_2F_1\left[\begin{matrix} \lambda - n - \frac{3}{4} \pm m \\ \lambda - n - k - \frac{1}{4} \end{matrix} ;\, \tfrac{1}{2}\right] {}_2F_1\left[\begin{matrix} \nu - \lambda + n \pm m_1 + 9/4 \\ \nu - \lambda + n - k_1 + 11/4 \end{matrix} ;\, \tfrac{1}{2}\right]$$

provided $R(\lambda + \frac{1}{4} \pm m) > R(n+1) > 0, \quad R(p) > 0,$

$$R(\nu \pm m \pm m_1 + 3/2) > 0 \quad \text{and} \quad R(\nu - \lambda + n \pm m_1 + 9/4) > 0.$$

My best thanks are due to Principal S. P. Chakravarti for the encouragement and facilities he gave to me, and to Dr. N. G. Shabde for his help and guidance during the preparation of this paper.

DEPARTMENT OF MATHEMATICS,
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JABALPUR.

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CALCUTTA MATHEMATICAL SOCIETY

Report of the Council for the year 1955 to the Annual General Meeting of the Society.

The Council of Calcutta Mathematical Society has the pleasure to submit the following report on the general concerns of the Society for the year 1955 as required by the provisions of Rule 25 of the Society's constitution.

The Council. The Council of the Society for the year 1955 consisting of the officers and other members elected at the last Annual General Meeting and co-opted thereafter, together with the Editorial Secretary, was constituted as follows:

President

S. K. Banerji,

Vice-Presidents

V. V. Narlikar,	B. R. Seth,
S. Gupta,	R. N. Sen,

S. Ghosh, (upto June, 1955).

Treasurer

S. C. Ghosh,

Secretary

U. R. Burman,

Editorial Secretary

P. K. Ghosh,

Other members of the Council

S. C. Mitra,	H. M. Sengupta,
M. Dutta,	B. N. Prosad,
M. Ray,	S. K. Chakraborty,
A. C. Choudhury,	N. N. Ghosh,
N. R. Sen,	V. P. Venkatachari,
N. L. Ghosh,	M. C. Chaki.

General. The various activities of the Society have been carried on throughout the year in much the same form as in the past few years. The Council has the pleasure to report that in January, 1955 on an invitation from the Society Prof. P. A. M. Dirac, Lucasian Professor of Mathematics in the University of Cambridge visited the Society and addressed the members. This visit of Prof. Dirac to this centre of research afforded an opportunity to the various workers to derive the benefit of discussions at close quarters with him, on problems of mutual interest The Society also arranged two other lectures by distinguished foreign scientists during the year under review. Prof. A. Walter, of the Institute of Applied Mathematics, University of Darmstadt, Germany delivered a lecture on "New Researches in Applied Mathematics" and Prof. Deuring, Director of the Mathematical Institute, Göttingen, Germany, delivered a lecture on "Recent Developments in Algebra and Number Theory".

The Council conveys its grateful thanks to the distinguished scientists for their illuminating lectures.

Membership. The Council reports with a profound sense of sorrow the loss of one of the Society's most celebrated Honorary Members, in the sudden death of Prof. Albert Einstein, of the Institute of Advanced Studies, Princeton, U. S. A. The death of Prof. Einstein removes one of the greatest thinkers the world has ever produced and has been felt in all parts of the world as something more than a public calamity.

The Council also mourns the sudden death of Prof. T. Vijayraghavan, Director of the Ramanujan Institute of Mathematics, Madras. The late Prof. Vijayraghavan was a Life member and a former Vice-President of the Society and was distinguished for his manifold contributions in the field of Pure Mathematics.

The Council regrets to report the loss of yet another member of the Society in the sudden death of Prof. V. P. Venkatachari, Head of the Department of Mathematics, Osmania University, Hyderabad (Deccan).

The Council desires to report that 13 new names have been added to the Society's list of ordinary members.

Meetings during 1955. The Council held 4 meetings during the year and there were 6 ordinary general meetings devoted to the reading of original papers communicated to the Society for publication in its Bulletin.

Publications During the year under review the Society has published three issues of the Bulletin, namely Vol 46, Nos. 3, 4 and Vol. 47, No. 1. The Council takes the opportunity to record here the Society's indebtedness to the authorities of the Calcutta University for printing the Bulletin free of charge and to the Officers and members of the staff of the University Press for their valued services.

Exchange of publications. The transmission of the Society's publications to various countries in the world has been carried on regularly during the year and 8 new exchange relations have also been established.

Library. It is gratifying to note that there has been no break during the year under review in the run of periodicals which the Society has been subscribing for the last few years.

Finance. The accounts of the Society for the year under review have been presented to the Council in the standardised form by the auditors Dr. T. C. Roy and Mr. K. M. Ghosh. The Council takes the opportunity to acknowledge its indebtedness to the Auditors for their honorary services.

The Society received the following grants during the year under review:

(i)	Government of West Bengal	Rs. 2,000/-
(ii)	National Institutes of Science of India	Rs. 500/-

and the Council offers its grateful thanks to the Government of West Bengal and the National Institute of Sciences of India for these grants.

CALCUTTA MATHEMATICAL SOCIETY

RECEIPTS AND DISBURSEMENTS ACCOUNTS OF THE CALCUTTA MATHEMATICAL SOCIETY FOR THE YEAR ENDING 31ST. DECEMBER, 1955.

Receipts

		Rs. as. p.	Rs. as p.
1. Opening Balance			
(a) With Secretary			
(i) In cash ...	0 0 9		
(ii) In stamps ...	0 0 6	0 1 3	
(b) Balance at Banks			
(i) State Bank of India (Gen. & C. E. Cullis Fund)	9,889 3 3		
(ii) Do (K. K. G. P. Fund)	754 9 8		
(iii) United Bank of India ...	968 5 0		
Do (in suspense) ...	36 0 0		
(iv) Postal Savings Banks ...	1,025 4 0	12,673 5 11	
(c) G. P. Notes (General Fund) (Face value Rs 6,000)		5,663 11 6	
(d) G. P. Notes (K. K. G. P Fund) (Face value Rs. 2,000)		1,937 7 9	20,274 10 5
2. Membership subscription ...	948 2 0		
3. Admission fees ...	110 0 0		
4. Sale proceeds ...	1,803 5 6	2,861 7 6	
5. Grants			
(i) Govt. of West Bengal ...	2,000 0 0		
(ii) Nat. Inst. of Sciences. ...	500 0 0	2,500 0 0	
6. Interest			
(a) G. P. Notes (General Fund)	180 0 0		
(b) Do (K. K. G. P. Fund) ...	60 0 0		
(c) Postal Savings Bank ...	14 15 0		
(d) 3 p.c. conversion Loan 1946/86 (C. E. Cullis Fund) ..	177 0 0	431 15 0	5,793 6 6
		Total ... Rs.	26,068 0 11

Disbursements

		Rs as. p.	Rs. as. p.
1. Establishment		915 13 9	
2. Meetings		81 1 6	
3. Books & Journals (including binding)		1,959 11 0	
4. Bulletins			
(a) Compositors' Salary ...	1,381 7 0		
(b) Papers, Blocks, Types etc ..	746 11 6		
(c) Postage ...	365 8 0	2,493 10 6	
5, Printing & Stationery		126 0 0	
6. Postage (General)		131 15 0	
7. Bank charges		30 7 0	
8. Miscellaneous (including conveyance charges)		57 0 0	5,795 10 9
9. Closing Balance			
(a) With Secretary			
(i) In cash ...	2 0 3		
(ii) In stamps ...	0 9 6	2 9 9	
(b) Balance at Banks			
(i) State Bank of India (General Fund) ...	3,246 7 3		
(ii) Do (K. K. G. P Fund)	814 1 8		
(iii) United Bank of India ...	923 12 3		
Do (in suspense) ...	36 0 0		
(iv) Postal Savings ...	1,090 3 0	6,110 8 2	
(c) G. P Notes (General Fund) (Face value Rs 6,000)		5,663 11 6	
(d) Do (K. K G P. Fund) (Face value Rs 2,000)		1,937 7 9	
(e) 3 p.c. Conversion Loan 1946/86. (C. E Cullis Fund Face value Rs 8,000) (including Bank's commission of Rs 16-8-0).		6,558 1 0	20,272 6 2
		Total ... Rs.	26,068 0 11

To

The Members of the Calcutta Mathematical Society,

We have examined the above Balance Sheet with the Books and Vouchers relating thereto and certify it to be correctly drawn up therefrom and in accordance with the information and explanations given to us.

K. M GHOSH }
T. C. ROY } *Auditors.*

TWO DIMENSIONAL SOURCE OR SINK IN A COMPRESSIBLE FLUID

By

M. RAY, *Agra*

(*Received—March 4, 1955*).

1. **Introduction.** Problem considered here is that of a flow of a compressible fluid due to a source or sink. Taking the flow as two dimensional and the source or sink at the origin, there is a circle $r = r_1$ on and beyond which the velocity is uniform, so also are the density, pressure and temperature. Motion considered is therefore within this circle. It has not been found possible to solve the complete equation of motion and the equation of energy, taking into account the effect of dissipation. So we have neglected dissipation. Secondly solution has been found only for a particular value of the Prandtl number σ, namely for $\sigma = \frac{1}{2}$ and that also when $(u_1/u)^4$ has been neglected where u_1 is the velocity of the outside stream, u the radial velocity.

The result has been expressed in terms of the Mach number of the outside stream and the case of an incompressible liquid for which the Mach number is infinite has been deduced from this.

2. **Basic equations.** We consider a motion in two dimensions having a central symmetry and converging towards or diverging from the centre. Take the centre of symmetry as origin and u be the radial velocity. Since all quantities are functions of r alone, the components of strain are given by

$$e_{rr} = 2\frac{du}{dr}, \quad e_{\theta\theta} = 2\frac{u}{r}, \quad e_{r\theta} = 0. \tag{1}$$

The divergence is

$$\Delta = \frac{du}{dr} + \frac{u}{r}. \tag{2}$$

The stress-components are given by

$$\left.\begin{aligned} p_{rr} &= -p + \lambda\Delta + \mu e_{rr} = -p + \lambda\left(\frac{du}{dr} + \frac{u}{r}\right) + 2\mu\frac{du}{dr}, \\ p_{\theta\theta} &= -p + \lambda\Delta + \mu e_{\theta\theta} = -p + \lambda\left(\frac{du}{dr} + \frac{u}{r}\right) + 2\mu\frac{u}{r}, \end{aligned}\right\} \tag{3}$$

$$p_{r\theta} = 0,$$

where p is the pressure.

With Stoke's assumption

$$\lambda + \tfrac{2}{3}\mu = 0, \tag{4}$$

we get

$$\left.\begin{aligned} p_{rr} &= -p+\tfrac{2}{3}\mu\left(2\frac{du}{dr}-\frac{u}{r}\right), \\ p_{rr}-p_{\theta\theta} &= 2\mu\left(\frac{du}{dr}-\frac{u}{r}\right) \end{aligned}\right\} \tag{5}$$

The equation of motion is

$$\frac{d}{dr}p_{rr}+\frac{1}{r}(p_{rr}-p_{\theta\theta}) = \varrho u\frac{du}{dr}, \tag{6}$$

where ϱ is the density, and the equation of continuity is

$$\frac{d}{dr}(\varrho ur) = 0$$

i.e.,

$$\varrho ur = \text{const.} = c. \tag{7}$$

With the help of (5) and (7), the equation (6) becomes

$$-r\frac{dp}{dr}+\tfrac{2}{3}r\frac{d}{dr}\left\{\mu\left(2\frac{du}{dr}-\frac{u}{r}\right)\right\}+2\mu\left(\frac{du}{dr}-\frac{u}{r}\right) = c\frac{du}{dr}. \tag{8}$$

If i be the enthalpy, we have for a perfect gas,

$$i\varrho = \frac{\gamma}{\gamma-1}p, \tag{9}$$

where $\gamma = c_p/c_v$, c_p, c_v being the specific heats at constant pressure and at constant volume and are assumed constant.

The equation of energy is

$$\varrho ur\frac{di}{dr} = \frac{1}{\sigma}\frac{d}{dr}\left(\mu r\frac{di}{dr}\right)+r\Phi, \tag{10}$$

where $\sigma = \mu c_p/\kappa$, called the Prandtl number and is taken as constant, μ the coefficient of viscosity, κ the thermal conductivity, and Φ the dissipation function, given by

$$4\Phi = \lambda(e_{rr}+e_{\theta\theta})^2+2\mu(e^2_{rr}+e^2_{\theta\theta}+2e^2_{r\theta})$$

or

$$\Phi = \tfrac{4}{3}\mu\left[\left(\frac{du}{dr}\right)^2+\left(\frac{u}{r}\right)^2-\frac{u}{r}\frac{du}{dr}\right]. \tag{11}$$

Further it is assumed that approximately

$$\frac{i}{i_1} = \frac{\mu}{\mu_1}, \tag{12}$$

where the suffix 1 denotes to the outside stream.

3. Source. Here the equation (7) becomes

$$\varrho ur = c = \varrho_1 u_1 r_1, \tag{13}$$

so neglecting dissipation, the equation (10) becomes

$$\sigma \varrho_1 u_1 r_1 \frac{di}{dr} = \frac{d}{dr}\left(\mu r \frac{di}{dr}\right). \tag{14}$$

Now from (9) and (12), we have

$$\mu = \frac{\mu_1}{i_1} i = \frac{\mu_1}{i_1} \cdot \frac{\gamma}{\gamma - 1} \frac{p}{\varrho}$$

$$= \frac{\mu_1 p u r}{p_1 u_1 r_1} = \frac{a}{u_1^2} p u r \tag{15}$$

where

$$a = \frac{\mu_1 u_1}{p_1 r_1}. \tag{16}$$

It is easy to find that the dimension of a is zero, so it is a pure number.

Hence the equation (14) becomes

$$\sigma \varrho_1 u_1 r_1 \frac{di}{dr} = \frac{d}{dr}\left(\frac{\mu_1}{i_1} i r \frac{di}{dr}\right). \tag{17}$$

A solution of this equation is

$$i = i_1 + b \log \frac{r}{r_1} \tag{18}$$

where

$$b = \sigma \frac{i_1 \varrho_1 u_1 r_1}{\mu_1} \tag{19}$$

i.e.,

$$b = \frac{\sigma \gamma}{\gamma - 1} \frac{u_1^2}{a}. \tag{20}$$

Thus

$$\mu = \mu_1 + \sigma \varrho_1 u_1 r_1 \log \frac{r}{r_1}. \tag{21}$$

Again from (15)

$$p u r = \frac{u_1^2}{a} \mu$$

$$= \frac{u_1^2}{a} \left(\mu_1 + \sigma \varrho_1 u_1 r_1 \log \frac{r}{r_1}\right). \tag{22}$$

Using (21) and (22) in the equation (8) which is

$$-r\frac{dp}{dr} + \tfrac{2}{3} r \frac{d}{dr}\left\{\mu\left(2\frac{du}{dr} - \frac{u}{r}\right) + 2\mu\left(\frac{du}{dr} - \frac{u}{r}\right)\right. = \varrho_1 u_1 r_1 \frac{du}{dr}, \tag{23}$$

we get

$$\frac{u_1^2}{a}\left(u+r\frac{du}{dr}\right)\left(\mu_1+\sigma\varrho_1 u_1 r_1\log\frac{r}{r_1}\right)-\sigma\frac{\varrho_1 u_1^3 r_1}{a}u$$

$$+\tfrac{2}{3}u^2r^2\left(\mu_1+\sigma\varrho_1 u_1 r_1\log\frac{r}{r_1}\right)\frac{d}{dr}\left(2\frac{du}{dr}-\frac{u}{r}\right)+\tfrac{2}{3}u^2r^2\left(2\frac{du}{dr}-\frac{u}{r}\right)\frac{\sigma\varrho_1 u_1 r_1}{r}$$

$$+2u^2r\left(\mu_1+\sigma\varrho_1 u_1 r_1\log\frac{r}{r_1}\right)\left(\frac{du}{dr}-\frac{u}{r}\right)-\varrho_1 u_1 r_1 u^2 r\frac{du}{dr}=0. \tag{24}$$

It is difficult to find a solution of this equation. The equation however has been divided into two parts and each part has been put to zero The solutions from these parts have been made to correspond to each other under certain limitations.

So to solve equation (24), we take

$$-\sigma\frac{u_1^3}{a}+\tfrac{4}{3}\sigma ur\frac{du}{dr}-\tfrac{2}{3}\sigma u^2-ur\frac{du}{dr}=0, \tag{25}$$

and
$$\frac{u_1^2}{a}\left(u+r\frac{du}{dr}\right)+\tfrac{2}{3}u^2r^2\frac{d}{dr}\left(2\frac{du}{dr}-\frac{u}{r}\right)+2u^2r\left(\frac{du}{dr}-\frac{u}{r}\right)=0. \tag{26}$$

From (25) we have

$$(4\sigma-3)aur\frac{du}{dr}=\sigma(2au^2+3u_1^2), \tag{27}$$

the solution of which is

$$2au^2+3u_1^2=Ar^{4\sigma/(4\sigma-3)}, \tag{28}$$

where A is a constant.

Using du/dr from (27) in the equation (26), we get on simplification,

$$4(2\sigma-3)(2\sigma-1)a^2u^4-3(4\sigma-3)(2\sigma-1)au_1^2u^2+9\sigma u_1^4=0. \tag{29}$$

If we neglect $(u_1/u)^4$, the above equation can be satisfied by $\sigma=\frac{1}{2}$. (30)

With this value of σ, we get from (28)

$$(2au^2+3u_1^2)r^2=(2a+3)u_1^2r_1^2,\ \ldots \tag{31}$$

so that
$$\left(\frac{u_1}{u}\right)^2=\frac{2ar^2}{(2a+3)r_1^2-3r^2}.$$

If $(u_1/u)^2<1$ then

$$2ar^2<(2a+3)r_1^2-3r^2$$

giving $r<r_1$,

which is true.

This consideration may somewhat justify our assumption that $(u_1/u)^4$ is very small and hence can be neglected.

In this case we have

$$i = i_1 + \frac{i_1\varrho_1 u_1 r_1}{2\mu_1}\log\frac{r}{r_1}. \tag{32}$$

But $r/r_1 < 1$, hence i is less than i_1. Thus i decreases inwards as we go towards the source; hence the solution will fail when r is less than its value given by

$$2\mu_1 = \varrho_1 u_1 r_1 \log\frac{r_1}{r}$$

$$= c\log\frac{r_1}{r}, \tag{33}$$

It is to be noted that $2\mu_1/c$ is a non-dimensional quantity hence a numerical constant.

The equation (33) gives the inner boundary *i.e.*, the motion is considered within the region bounded by (33) and $r = r_1$.

We also note that the numerical constant a can be expressed in terms of the Mach number M of the outside stream,

for $a = \mu_1 u_1/(p_1 r_1)$ and $u_1^2/i_1 = (\gamma-1)M^2$

hence $a = \mu_1\gamma M^2/c$. (34)

4. **Sink.** Here the corresponding equations are

$$\varrho u r = -c = -\varrho_1 u_1 r_1 \tag{13a}$$

so that $c = \varrho_1 u_1 r_1$. (15a)

Also $\mu = -\dfrac{a}{u_1^2}\,pur$

where as before $a = \mu_1 u_1/(p_1 r_1)$. (16a)

Corresponding to (14), we have

$$-\sigma\varrho_1 u_1 r_1\frac{di}{dr} = \frac{d}{dr}\left(\mu r\frac{di}{dr}\right) \tag{14a}$$

solution of which is $i = i_1 + b\log\dfrac{r}{r_1}$, (18a)

where $b = -\dfrac{\sigma i_1\varrho_1 u_1 r_1}{\mu_1}$ (19a)

$$= -\frac{\sigma\gamma}{\gamma-1}\,\frac{u_1^2}{a}. \tag{20a}$$

Also
$$\mu = \mu_1 - \sigma\varrho_1 u_1 r_1 \log\frac{r}{r_1}. \tag{21a}$$

The equations (23) and (24) here become

$$-r\frac{dp}{dr} + \tfrac{2}{3}r\frac{d}{dr}\left\{\mu\left(2\frac{du}{dr} - \frac{u}{r}\right)\right\} + 2\mu\left(\frac{du}{dr} - \frac{u}{r}\right) = -\varrho_1 u_1 r_1 \frac{du}{dr}, \tag{23a}$$

and
$$-\frac{u_1^2}{a}\left(u + r\frac{du}{dr}\right)\left(\mu_1 - \sigma\varrho_1 u_1 r_1 \log\frac{r}{r_1}\right) - \frac{\sigma\varrho_1 u_1^3 r_1}{a}u$$

$$+\tfrac{2}{3}u^2r^2\left(\mu_1 - \sigma\varrho_1 u_1 r_1 \log\frac{r}{r_1}\right)\frac{d}{dr}\left(2\frac{du}{dr} - \frac{u}{r}\right) - \tfrac{2}{3}u^2r^2\left(2\frac{du}{dr} - \frac{u}{r}\right)\frac{\sigma\varrho_1 u_1 r_1}{r}$$

$$+2u^2r\left(\mu_1 - \sigma\varrho_1 u_1 r_1 \log\frac{r}{r_1}\right)\left(\frac{du}{dr} - \frac{u}{r}\right) + \varrho_1 u_1 r_1 u^2 r\frac{du}{dr} = 0. \tag{24a}$$

The equation (28) becomes

$$2au^2 - 3u_1^2 = Ar^{4\sigma/(4\sigma-3)} \tag{28a}$$

and (29) becomes

$$4(2\sigma-3)(2\sigma-1)a^2u^4 + 3(4\sigma-3)(2\sigma-1)au_1^2u^2 + 9\sigma u_1^4 = 0. \tag{29a}$$

Hence again neglecting $(u_1/u)^4$, we get

$$\sigma = \tfrac{1}{2} \tag{30a}$$

and
$$(2au^2 - 3u_1^2)r^2 = (2a-3)u_1^2r_1^2. \tag{31a}$$

Here
$$i = i_1 - \frac{i_1\varrho_1 u_1 r_1}{2\mu_1}\log\frac{r}{r_1}. \tag{32a}$$

so that i increases towards the sink.

5. **Incompressible liquid.** In the degenerate case when $i = i_1$ so that $b = 0$ hence a is infinite.

Here from (31) we have

$$\left(2u^2 + \frac{3u_1^2}{a}\right)r^2 = \left(2 + \frac{3}{a}\right)u_1^2r_1^2$$

or when a is infinite, we get

$$u^2r^2 = u_1^2r_1^2$$

or
$$ur = \pm u_1 r_1 \tag{35}$$

This corresponds to the case of an incompressible liquid.

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ELASTIC STRESS DISTRIBUTION IN A THREE-LAYERED SYSTEM DUE TO A CONCENTRATED FORCE*

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Summary. The problem of the stress distribution in a semi-infinite mass having two layers of different material over it due to a concentrated surface has been solved by the use of Hankel transform of the stress function. A formula for the "Equivalent Depth" replacing the two upper layers by the same material as the underlying mass has been derived. Numerical results have been considered for a particular case.

1. Introduction. The design of highways and airport runways as well as the foundation problems is soil mechanics, specially when the earth mass supporting a heavy structure has different soil strata over it, lead to the investigation of the stress distribution in a layered system. Attempts to obtain a sufficiently close approximation to the actual physical conditions encountered was made by Burmister (1943, 1945) in a series of papers. Fox (1948) extended Burmister's work by computing the stresses in the lower layer of the two-layer system. He also computed in the same paper stresses in both layers, using the "Relaxation Method" developed by Southwell. In a latter paper (1951), Acum and Fox attacked a problem in a three-alyer system only by the method of Burmister; for, in that case, Relaxation Method was found to be laborious. In the present paper, a three-layer system has been considered when a concentrated force acts over the upper layer. The method of solution consists in taking the Hankel transform of the stress function instead of the stress function itself. A formula for the "Equivalent Depth" replacing the two upper layers by the same material as the underlying mass extending to a great extent, is derived in terms of a certain integral involving determinants and thus the whole problem is reduced to the Boussinesq problem of a concentrated force on a semi-infinite mass. Numerical results are considered when the upper layer consists of concrete pavement, the second layer is of gravel base course and the lower mass is the natural soil, as in the case of airport runways.

2. Statement of the problem. Let the material under consideration occupy the lower half of the plane $z = 0$ and the axis of z be taken positive when drawn into the material. Let the lower boundaries of the first and the second layers be given by the planes $z = h_1$ and $z = h_2$ respectively and the underlying mass extend to infinity. The interfaces are supposed to be perfectly rough so that the stresses and the displacements are continuous across it. It is required to find the stresses at any point in the material with chief emphasis on stresses in the underlying mass, when a concentration force P acts at the origin in the positive direction of the z-axis.

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3. Method of solution. Since the physical quantities involved in the problem are all symmetrical about the z-axis, the four non-zero stress components σ_z, σ_r, σ_θ and τ_{zr} as well as the two non-zero displacement components u_r and u_z are obtained in terms of a stress function φ satisfying the differential equation (Fox, 1948, with different notation for the stresses and the displacements)

$$\nabla^4\varphi \equiv \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right)^2 \varphi = 0 \tag{1}$$

where $r = (x^2+y^2)^{\frac{1}{2}}$ is the distance of a point (x, y, z) from the z-axis. They are given by the following expressions:

$$\sigma_z = \frac{\partial}{\partial z}\left[(2-\sigma)\nabla^2\varphi - \frac{\partial^2\varphi}{\partial z^2} \right] \tag{2}$$

$$\sigma_r = \frac{\partial}{\partial z}\left[\sigma\nabla^2\varphi - \frac{\partial^2\varphi}{\partial r^2} \right] \tag{3}$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left[\sigma\nabla^2\varphi - \frac{1}{r}\frac{\partial\varphi}{\partial r} \right] \tag{4}$$

$$\tau_{zr} = \frac{\partial}{\partial r}\left[(1-\sigma)\nabla^2\varphi - \frac{\partial^2\varphi}{\partial z^2} \right] \tag{5}$$

$$u_r = -\frac{1+\sigma}{E}\frac{\partial^2\varphi}{\partial r\,\partial z} \tag{6}$$

$$u_z = \frac{1+\sigma}{E}\left[(1-2\sigma)\nabla^2\varphi + \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} \right] \tag{7}$$

In the present discussion, the stress function φ will be replaced by its Hankel transform G, defined by,

$$G(\xi, z) = \int_0^\infty r\varphi J_0(\xi r)\,dr, \ (\xi > 0, z > 0) \tag{8}$$

where J_0 denotes the Bessel function of order zero.

Multiplying both sides of equations (1), (2) and (7) by $rJ_0(\xi r)$ and equations (5) and (6) by $rJ_1(\xi r)$, integrating over r from 0 to ∞ and using the results (Sneddon, 1951)

$$\int_0^\infty r\frac{df}{dr}J_1(\xi r)\,dr = -\xi\bar{f}(\xi)$$

$$\int_0^\infty r\left(\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} \right) J_0(\xi r)\,dr = -\xi^2\bar{f}(\xi)$$

where, $$\bar{f}(\xi) = \int_0^\infty rf(r)J_0(\xi r)\,dr$$

we obtain,

$$\left(\frac{d^2}{dz^2}-\xi^2\right)^2 G(\xi, z)=0 \tag{9}$$

$$\int_0^\infty r\sigma_z J_0(\xi r)dr=(1-\sigma)\frac{d^3G}{dz^3}-(2-\sigma)\xi^2\frac{dG}{dz} \tag{10}$$

$$\int_0^\infty r\tau_{zr} J_1(\xi r)\,dr=\xi\left[\sigma\frac{d^2G}{dz^2}+(1-\sigma)\xi^2 G\right] \tag{11}$$

$$\int_0^\infty ru_r J_1(\xi r)dr=\frac{1+\sigma}{E}\xi\frac{dG}{dz} \tag{12}$$

$$\int_0^\infty ru_z J_0(\xi r)dr=\frac{(1+\sigma)}{E}\left[(1-2\sigma)\frac{d^2G}{dz^2}-2(1-\sigma)\xi^2 G\right] \tag{13}$$

Using Hankel Inverse Transform (Sneddon, 1951), we get from equations (10) to (13), the expressions for stresses and displacements as

$$\sigma_z=\int_0^\infty \xi\left[(1-\sigma)\frac{d^3G}{dz^3}-(2-\sigma)\xi^2\frac{dG}{dZ}\right]J_0(\xi r)d\xi \tag{14}$$

$$\tau_{zr}=\int_0^\infty \xi^2\left[\sigma\frac{d^2G}{dz^2}+(1-\sigma)\xi^2 G\right]J_1(\xi r)d\xi \tag{15}$$

$$u_r=\frac{(1+\sigma)}{E}\int_0^\infty \xi^2\frac{dG}{dZ}J_1(\xi r)d\xi \tag{16}$$

$$u_z=\frac{(1+\sigma)}{E}\int_0^\infty \xi\left[(1-2\sigma)\frac{d^2G}{dz^2}-2(1-\sigma)\xi^2 G\right]J_0(\xi r)d\xi \tag{17}$$

The stresses σ_r and σ_θ can also be expressed in terms of G by the method given by Sneddon

The solution of the equation (9) is given by

$$G(\xi, z)=(A+Bz)\cosh\xi z+(C+Dz)\sinh\xi z,\ (\xi\geq 0,\ z\geq 0)$$

where A, B, C and D are functions of ξ to be determined from the boundary conditions.

4. Solution of the the problem. We shall use the suffixes 1, 2 and 3 for the quantities connected with different layers. Thus E_1, E_2 and E_3 represent the elastic constants for the upper layer, middle layer and the underlying mass respectively and similar notations will be used for other physical constants and assumed functions.

The concentrated force P acting at the origin on the upper surface can be looked upon as an uniformly distributed pressure $P/(\pi a^2)$ over a small circular area of radius 'a' surrounding the origin. Thus at

$$\left.\begin{aligned} z = 0, \quad \sigma_z &= -P/(\pi a^2) \qquad 0 \leq r \leq a \\ &= 0 \qquad\qquad\quad r > a \end{aligned}\right\} \tag{19}$$

where $a \to 0$

The expressions for $G(\xi, z)$ for different layers are as follows:—

For the upper layer, $G_1(\xi, z) = (A_1 + B_1 z)\cosh\xi z + (C_1 + D_1 z)\sinh\xi z$ (20)

For the middle layer, $G_2 = (A_2 + B_2 z)\cosh\xi z + (C_2 + D_2 z)\sinh\xi z$ (21)

For the underlying mass, $G_3 = (A_3 + B_3 z)e^{-\xi z}$ (22)

where $\xi \geq 0, \quad z \geq 0.$

The fact that the stresses and the displacements in the underlying mass vanish as z tends to infinity has been considered in writting the expression for G_3. The boundary conditions are as follows:—

At $z = 0$, $(\sigma_{z1})_{z=0}$ is given by (19)

$$(\tau_{zr1})_{z=0} = 0 \tag{23}$$

At $z = h_1$,

$$(\sigma_{z1})_{z=h_1} = (\sigma_{z2})_{z=h_1} \tag{24}$$

$$(\tau_{zr1})_{z=h_1} = (\tau_{zr2})_{z=h_1} \tag{25}$$

$$(u_{r1})_{z=h_1} = (u_{r2})_{z=h_1} \tag{26}$$

$$(u_{z1})_{z=h_1} = (u_{z2})_{z=h_1} \tag{27}$$

At $z = h_2$

$$(\sigma_{z2})_{z=h_2} = (\sigma_{z3})_{z=h_2} \tag{28}$$

$$(\tau_{zr2})_{z=h_2} = (\tau_{zr3})_{z=h_2} \tag{29}$$

$$(u_{r2})_{z=h_2} = (u_{r3})_{z=h_2} \tag{31}$$

$$(u_{z2})_{z=h_2} = (u_{z3})_{z=h_2} \tag{31}$$

The ten equations (19), (23)—(31) will determine the ten unknown functions $A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, A_3$ and B_3.

Substituting the value of G_1 from (20) in (10) and using the condition (19), we have,

$$-\xi^2[-B_1(1-2\sigma_1) + C_1\xi] = \lim_{a\to 0}\int_0^a r\left(-\frac{P}{\pi a^2}\right)J_0(\xi r)dr$$

giving,

$$C_1\xi = B_1(1-2\sigma_1) + P/(2\pi\xi^2) \tag{32}$$

Similarly, from (20). (11) and (23), we get,

$$A_1\xi + 2\sigma_1 D_1 = 0 \tag{33}$$

Substituting the values of G_1, G_2 and G_3 from (20) (21) and (22) in (10)—(13) and using the conditions (24)—(31) and eliminating A_1 and C_1 with the help of (32) and (33), we get on making the substitution,

$$\left.\begin{array}{l} h_1/h_2 = t, \quad h_2\xi = \eta, \quad A_2/h_2 = = A'_2, \quad C_2/h_2 = C'_2 \quad A_3/h_2 = h_2 = A'_3 \\ p_1 = \cosh \xi h_1, \quad q_1 = \sinh \xi h_1, \quad p_2 = \cosh \xi h_2, \quad q_2 = \sinh \xi h_2 \\ (1+\sigma_1)/(1+\sigma_2) = lE_1/E_2, \quad (1+\sigma_3)/(1+\sigma_2) = mE_2/E_3 \end{array}\right\} \quad (34)$$

the equations connecting the unknowns,

$$A'_2\eta q_1 + B_2\{t\eta q_1 - (1-2\sigma_2)p_1\} + C'_2\eta p_1 + D_2\{t\eta p_1 - (1-2\sigma_2)q_1\} - B_1 t\eta q_1$$
$$- D_1\{t\eta p_1 - q_1\} - \frac{Ph_2^2}{2\pi}.\frac{p_1}{\eta^2} = 0 \quad (35)$$

$$A'_2\eta p_1 + B_2\{t\eta p_1 + 2\sigma_2 q_1\} + C'_2\eta q_1 + D_2\{t\eta q_1 + 2\sigma_2 p_1\} - B_1(t\eta p_1 + q_1)$$
$$- D_1 t\eta q_1 - \frac{Ph_2^2}{2\pi}\,\frac{q_1}{\eta^2} = 0 \quad (36)$$

$$A'_2\eta q_1 + B_2(t\eta q_1 + p_1) + C'_2\eta p_1 + D_2\{t\eta p_1 + q_1\}$$
$$- l\left[B_1\{t\eta q_1 + 2(1-\sigma_1)p_1\} + D_1\{t\eta p_1 + (1-2\sigma_1)q_1\} + \frac{Ph_2^2}{2\pi}.\frac{p_1}{\eta^2}\right] = 0 \quad (37)$$

$$A'_2\eta p_1 + B_2\{t\eta p_1 - 2(1-2\sigma_2)q_1\} + C'_2\eta q_1 + D_2\{t\eta q_1 - 2(1-2\sigma_2)p_1\}$$
$$- l\left[B_1\{t\eta p_1 - (1-2\sigma_1)q_1\} + D_1\{t\eta q_1 - 2(1-\sigma_1)p_1\} + \frac{Ph_2^2}{2\pi}.\frac{q_1}{\eta^2}\right] = 0 \quad (38)$$

$$A'_2\eta q_1 + B_2\{\eta q_2 - (1-2\sigma_2)p_2\} + C'_2\eta p_2 + D_2\{\eta p_2 - (1-2\sigma_2)q_2\}$$
$$+ e^{-\eta}[A'_3\eta + B_3(\eta + 1 - 2\sigma_3)] = 0 \quad (39)$$

$$A'_2\eta p_2 + B_2(\eta p_2 + 2\sigma_2 q_2) + C'_2\eta q_2 + D_2(\eta q_2 + 2\sigma_2 p_2) - e^{-\eta}[A'_3\eta + B_3(\eta - 2\sigma_3)] = 0 \quad (40)$$

$$A'_2\eta q_2 + B_2(p_2 + \eta q_2) + C'_2\eta p_2 + D_2(q_2 + \eta p_2) + me^{-\eta}[A'_3\eta + B_3(\eta - 1)] = 0 \quad (41)$$

$$A'_2\eta p_2 + B_2\{\eta p_2 - 2(1-2\sigma_2)q_2\} + C'_2\eta q_2 + D_2\{\eta q_2 - 2(1-2\sigma_2)p_2\}$$
$$- me^{-\eta}[A'_3\eta + B_3\{\eta + 2(1-2\sigma_3)\}] = 0 \quad (42)$$

The equations (35)—(42) give theoratically the eight unknown function which when substituted in relations (14)—(17) give the stresses and displacements.

5. Numerical evaluation. We suppose that the upper layer consists of the concrete pavement, the second layer is of gravel base-course and the underlying mass the natural soil and we take the following elastic constants for the different material.

For the upper layer, $E_1 = 3{\cdot}0 \times 10^6$ lbs. per. sq. inch. $\sigma_1 = 0{\cdot}15$

For the second layer, $E_2 = 1{\cdot}5 \times 10^6$ lbs. per. sq. inch. $\sigma_2 = 0{\cdot}25$

For the lower mass, $E_3 = 0{\cdot}6 \times 10^6$ lbs. per. sq. inch. $\sigma_2 = 0{\cdot}50$

Thus $E_1 : E_1 : E_2 = 10 : 5 : 2. \quad l = 0{\cdot}46, \quad m = 3{\cdot}0. \quad (43)$

Then the equations (35)—(42) reduce to the following simplified forms.

$$A'_2\eta q_1 + B_2(t\eta q_1 - 0{\cdot}5p_1) + C'_2\eta p_1 + D_2(t\eta p_1 - 0{\cdot}5q_1) - B_1 t\eta q_1$$
$$- D_1(t\eta p_1 - q_1) - \frac{Ph_2^2}{2\pi}.\frac{p_1}{\eta^2} = 0 \quad (44)$$

$$A'_2\eta p_1 \pm B_2(t\eta p_1+0\cdot5q_1)+C'_2\eta q_1+D_3(t\eta q_1+0\cdot5p_1)-B_1(t\eta p_1+q_1)$$
$$-D_1 t\eta q_1-\frac{Ph_2^2}{2\pi}.\frac{q_1}{\eta^2}=0 \qquad (45)$$

$$A'_2\eta q_1+B_2(t\eta q_1+p_1)+C'_2\eta p_1+D_2(t\eta p_1+q_1)$$
$$-0\cdot46\left[B_1(t\eta q_1+1\cdot7p_1+D_1(t\eta p_1+0\cdot7q_1)+\frac{Ph_3^2}{2\pi}.\frac{p_1}{\eta^2}\right]=0 \qquad (46)$$

$$A'_2\eta p_1+B_2(t\eta p_1-q_1)+C'_2\eta q_1+D_2(t\eta q_1-p_1)$$
$$-0\cdot46\left[B_1(t\eta p_1-0\cdot7q_1)+D_1(t\eta q_1-1\cdot7p_1)+\frac{Ph_2^2}{2\pi}.\frac{q_1}{\eta^2}\right]=0 \qquad (47)$$

$$A'_2\eta q_2+B_2(\eta q_2-0\cdot5p_2)+C'_2\eta p_2+D_2(\eta p_2-0\cdot5q_2)+e^{-\eta}[A'_3\eta+B_3\eta]=0 \qquad (48)$$

$$A'_2\eta p_2+B_2(\eta p_2+0\cdot5q_2)+C'_2\eta q_2+D_2(\eta q_2+0\cdot5p_2)-e^{-\eta}[A'_3\eta+B_3(\eta-1)]=0 \qquad (49)$$

$$A'_2\eta q_2+B_2(p_2+\eta q_2)+C'_2\eta p_2+D_2(q_2+\eta p_2)+3\cdot0e^{-\eta}[A'_3\eta+B_3(\eta-1)]=0 \qquad (50)$$

$$A'_2\eta p_2+B_2(\eta p_2-q_2)+C'_2\eta q_2+D(\eta q_2-p_2)-3\cdot0e^{-\eta}[A'_3\eta+B_3\eta]=0 \qquad (51)$$

Equations (44)—(51) can be solved with the help of determinants. Thus

$$\frac{A'_2\eta}{\Delta_1}=-\frac{B_2}{\Delta_2}=\frac{C'_2\eta}{\Delta_3}=-\frac{D_2}{\Delta_4}=-\frac{B_1}{\Delta_5}=\frac{D_1}{\Delta_6}=\frac{A'_3\eta e^{-\eta}}{\Delta_7}=-\frac{B_3e^{-\eta}}{\Delta_8}=-\frac{Ph_2^2}{2\pi\eta^2}\frac{1}{\Delta_9} \qquad (52)$$

where the determinants Δ_1, Δ_2.. Δ_9 are each of eighth order The eight columns of the several determinants are formed of the coefficients of $A'_2\eta$, B_2, $C'_2\eta$, D_2, $-B_1$. $-D_1$, $A'_3\eta e^{-\eta}$, $B_3e^{-\eta}$ and $-\frac{Ph_2^2}{2\pi\eta^2}$ respectively, and the order of arrangement of these columns are the same as the order of arrangement of the quantities $A'_2\eta$,. ... in equation (52).

Thus

$$\Delta_9 \equiv \begin{vmatrix} q_1 & t\eta q_1-0\cdot5p_1 & p_1 & t\eta p_1-0\cdot5q_1 & t\eta q_1 & t\eta p_1-q_1 & 0 & 0 \\ p_1 & t\eta p_1+0\cdot5q_1 & q_1 & t\eta q_1+0\cdot5p_1 & t\eta p_1+q_1 & t\eta q_1 & 0 & 0 \\ q_1 & t\eta q_1+p_1 & p_1 & t\eta p_1+q_1 & 0\cdot46(t\eta q_1+1\cdot7p_1) & 0\cdot46(t\eta p_1+0\cdot7q_1) & 0 & 0 \\ p_1 & t\eta p_1-q_1 & q_1 & t\eta q_1-p_1 & 0\cdot46(t\eta p_1-0\cdot7q_1) & 0\cdot46(t\eta q_1-0\cdot7p_1) & 0 & 0 \\ q_2 & \eta q_2-0\cdot5p_2 & p_2 & \eta p_2-0\cdot5q_2 & 0 & 0 & 1\cdot0 & \eta \\ p_2 & \eta p_2+0\cdot5q_2 & q_2 & \eta q_2+0\cdot5p_2 & 0 & 0 & -1\cdot0 & -(\eta-1) \\ q_2 & p_2+\eta q_2 & p_2 & q_2+\eta p_2 & 0 & 0 & 3\cdot0 & 3(\eta-1) \\ p_2 & \eta p_2-q_2 & q_2 & \eta q_2-p_2 & 0 & 0 & -3\cdot0 & 3\eta \end{vmatrix} \qquad (53)$$

The determinant Δ_8 will be obtained from (53) by replacing the last column by a column whose first four elements are p_1, q_1, p_1 and q_1 respectively and the remaining four elements are zeroes. Similarly, Δ_7 will be obtained from (53) by removing the seventh column and adding at the extreme right, a column same as in the case of Δ_8 mentioned above.

Substituting the value of G_3 from (22) in (14) and eliminating the unknown functions with the help of (52), we obtain, when

$$r = 0, \quad z = h_2, \quad (\sigma_z)_{\substack{r=0\\ z=h_2}} = -PI(t)/(2\pi h_2^2) \tag{54}$$

where

$$I(t) \equiv \int_0^\infty \eta(\Delta_7 - \eta\Delta_8)/\Delta_0 d\eta \tag{55}$$

In a homogeneous mass where there are no layers the vertical pressure σ_z at a depth d_2 vertically under the load point (*i, e*, $r = 0$, $z = d_2$) is given by the Boussinesq formula (Krynine, 1947)

$$(\sigma_z)_{\substack{r=0\\ z=d_2}} = -3P/(2\pi d_2^2) \tag{56}$$

If d_2 be the "Equivalent depth" replacing the two upper layers by the same material as the lower mass, such that the stresses in the lower mass remain unchanged, equating (54) and (56) we have

$$d_2 = \sqrt{3}h_2 I^{-\frac{1}{2}}$$

When the Bearing Capacity (Krynine, 1947) of the soil is known, d_2 is obtained from the relation (56) for a given load P and then the relation (57) gives h_2. Thus the thickness of the pavement with relation to the base course is determined. The value of $I(t)$ can, of course. be determined from the relation (55) for different values of t. In practical problems, t is less than 0·5. If we take $t = 0{\cdot}5$, the values of I and d_2 are 2·7325 and 1·048 h_2 respectively.

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ON MEIJER TRANSFORM OF TWO VARIABLES

By

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I

Meijer (1941) introduced the transform

$$\varphi(p) = p\int_0^\infty e^{-\frac{1}{2}px} W_{k+\frac{1}{2},\, m}(px)\,(px)^{-k-\frac{1}{2}} f(x)\,dx,$$

where $W_{k,m}(z)$ is the Whittaker function. We shall denote this integral equation as in Jaiswal (1952) by

$$f(x)\xrightarrow[m]{k+\frac{1}{2}}\varphi(p).$$

Similarly, if x is replaced by y and p by q, we shall have

$$\varphi(q) = q\int_0^\infty e^{-\frac{1}{2}qy} W_{k+\frac{1}{2},\, m}(qy)(qy)^{-k-\frac{1}{2}} f(y)\,dy.$$

A function of two variables x and y, $f(x, y)$, will be replaced by a function of two variables corresponding to p and q, $\varphi(p, q)$, by the double integral relation*

$$\varphi(p, q) = pq\int_0^\infty\int e^{-\frac{1}{2}px} e^{-\frac{1}{2}qy} W_{k+\frac{1}{2},\, m}(px) W_{k_1+\frac{1}{2},\, m_1}(qy)\;(px)^{-k-\frac{1}{2}}(qy)^{-k_1-\frac{1}{2}} f(x, y)\,dx\,dy \quad (1),$$

which we shall denote symbolically by

$$f(x, y)\xrightarrow[m,\ m_1]{k+\frac{1}{2},\ k_1+\frac{1}{2}}\varphi(p, q).$$

When $k = \pm m$ and $k_1 = \pm m_1$, due to the identity

$$e^{-\frac{1}{2}px} \equiv (px)^{-m-\frac{1}{2}} W_{m+\frac{1}{2},\, m}(px)$$

(1) reduces to

$$\varphi(p, q) = pq\int_0^\infty\int e^{-px} e^{-qy} f(x, y)\,dx\,dy,$$

which is symbolically denoted as

$$\varphi(pq) \doteqdot f(x, y)$$

and is well known as Laplace Transform of two variables.

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* The symbol $\int_0^\infty\int$ denotes $\int_0^\infty\int_0^\infty$.

When $k = \frac{1}{2}n - \frac{1}{4}$, $m = \pm\frac{1}{4}$, $k_1 = \frac{1}{2}n_1 - \frac{1}{4}$ and $m_1 = \pm\frac{1}{4}$,
(1) reduces to

$$\varphi(p, q) = 2^{-\frac{1}{4}n - \frac{1}{4}n_1} pq \int_0^\infty\int e^{-\frac{1}{2}px - \frac{1}{2}qy}(px)^{-\frac{1}{4}n}(qy)^{-\frac{1}{4}n_1} D_n((2px)^{\frac{1}{2}})\, D_{n_1}((2qy)^{\frac{1}{2}})\, f(x, y)\, dx\, dy$$

and will be known as $D_n\, D_{n_1}$-transform and symbolically denoted as

$$f(x, y)\xrightarrow[\pm\frac{1}{4},\ \pm\frac{1}{4}]{\frac{1}{2}n + \frac{1}{4},\ \frac{1}{2}n_1 + \frac{1}{4}}\varphi(p, q).$$

When $k = \frac{1}{2}n + l$, $m = \pm\frac{1}{2}n$, $k_1 = \frac{1}{2}n_1 + l_1$ and $m_1 = \pm\frac{1}{2}n_1$,
(1) reduces to

$$\varphi(p, q) = (-)^{l+l_1}\, l\,!\; l_1\,! \int_0^\infty\int e^{-px-qy}(px)^{-l}(py)^{-l_1}L^{n}{}_{l}(px)L^{n_1}{}_{l_1}(qy)\, f(x, y)\, dx dy,$$

where l and l_1 are positive integers and will be known as $L^{n}{}_{l}\, L^{n_1}{}_{l_1}$ transform and symbolically denoted as

$$f(x, y)\xrightarrow[\pm\frac{1}{2}n,\ \pm\frac{1}{2}n_1]{\frac{1}{2}n + l + \frac{1}{2},\ \frac{1}{2}n_1 + l_1 + \frac{1}{2}}\varphi(p, q).$$

The object of this paper is to develop the theory of Meijer transform of two variables on parallel lines to that of one variable. In section I we have given some of the rules of Meijer transform. Since they are similar to those of Whittaker transform [Bose (1952)], therefore, we have only given those that we have used or the ones that are new In section II we have found out the Meijer transforms of some of the known functions of Analysis, which have been latter on utilized in illustrating the theorems. In section III we have obtained certain results by utilizing the properties of Generalized Laguerre polynomial, Parabolic cylinder function and Bateman's k-function.

(1) If

$$f(x, y) = f_1(x)\, f_2(y)$$

and if

$$f_1(x)\xrightarrow[m]{k+\frac{1}{2}}\varphi_1(p)$$

$$f_2(x)\xrightarrow[m]{k+\frac{1}{2}}\varphi_2(p),$$

then

$$\varphi(p, q) = \varphi_1(p)\, \varphi_2(q).$$

(2) If

$$\varphi_1(p, q)\xrightarrow[m,\ m_1]{k+\frac{1}{2},\ k_1+\frac{1}{2}}f_1(x, y)$$

and

$$\varphi_2(p, q)\xrightarrow[m,\ m_1]{k+\frac{1}{2},\ k_1+\frac{1}{2}}f_2(x, y),$$

then

$$\int_0^\infty\int \varphi_1(, u)vf_2(u, v)\, \frac{du}{u}\ \frac{dv}{v} = \int_0^\infty\int \varphi_2(s, t)\, f_1(s, t)\frac{ds}{s}\frac{dt}{t},$$

provided the integrals are absolutely convergent.

From the definition, we have

$$\varphi_1(u, v) = uv \int_0^\infty \int e^{-\frac{1}{2}us-\frac{1}{2}vt} W_{k+\frac{1}{2},\, m}(us)\, W_{k_1+\frac{1}{2},\, m_1}(vt)\, (us)^{-k-\frac{1}{2}} (vt)^{-k_1-\frac{1}{2}} f_1(s\ t)\, ds dt \quad (1.1)$$

$$\varphi_2(t,\ )\, s = ts \int_0^\infty \int e^{-\frac{1}{2}su-\frac{1}{2}tv}\, W_{k+\frac{1}{2},\, m}(su)\, W_{k_1+\frac{1}{2},\, m_1}(tv)\, (su)^{-k-\frac{1}{2}} (tv)^{-k_1-\frac{1}{2}} \times$$

$$f_2(u, v) du dv \qquad (1.2)$$

Now

$$\int_0^\infty \int \varphi_1(u, v) f_2(u, v) \frac{du}{u} \frac{dv}{v} = \int_0^\infty \int f_2(u, v)\, du dv \Big[\int_0^\infty \int e^{-\frac{1}{2}us-\frac{1}{2}vt} W_{k+\frac{1}{2},\, m} \times$$

$$(us) W_{k_1+\frac{1}{2},\, m_1}(vt) . (us)^{-k-\frac{1}{2}} (vt)^{-k_1-\frac{1}{2}} f_1(s,t)\, ds dt \Big]$$

$$= \int_0^\infty \int f_1(,\, st)\, ds dt \Big[\int_0^\infty \int e^{-\frac{1}{2}us-\frac{1}{2}vt} W_{k+\frac{1}{2},\, m}(us) W_{k_1+\frac{1}{2},\, m_1}(vt)\, (us)^{-k-\frac{1}{2}} \times$$

$$(vt)^{-k_1-\frac{1}{2}} f_2(uv)\, du dv \Big]$$

$$= \int_0^\infty \int f_1(s, t)\, \varphi_2(s, t) \frac{ds\, dt}{st}, \quad \text{from (1.2)}$$

provided the change of order of integration is justified.

When $k = \pm m$ and $k_1 = \pm m_1$, this rule reduces to a theorem of Bose (1949, p. 173)

II

In this section we have obtained the Meijer transform of some of the known functions.

(*a*) If
$$f(x, y) = x^n y^{n_1}$$
then*
$$\varphi(p, q) = \frac{\Gamma_*(n-k+1 \pm m)}{\Gamma(n-2k+1)} \frac{\Gamma_*(n_1-k_1+1 \pm m_1)}{\Gamma(n_1-2k_1+1)} p^{-n} q^{-n_1},$$

provided $R(n-k+1 \pm m) > 0$, $R(n_1-k_1+1 \pm m_1) > 0$ and $R(p, q) \geq 0$.

If $k = \pm m$ and $k_1 = \pm m_1$, we have [McLachlan and Humbert, 1941, p. 62]

$$x^n y^n \doteqdot \Gamma(n+1) \Gamma(n_1+1) p^{-n} q^{-n_1},$$

$R(n+1) > 0$, $R(n_1+1) > 0$ and $R(p, q) > 0$.

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[*] The symbol $\Gamma_+(\alpha \pm \beta)$ denotes $\Gamma(\alpha+\beta)\,\Gamma(\alpha-\beta)$ and $R(p, q) > 0$ denotes $R(p) > 0$ and $R(q) > 0$. Also ${}_2F_1\left[\begin{matrix}\alpha \pm \beta \\ \gamma\end{matrix} ; x\right]$ denotes ${}_2F_1\left[\begin{matrix}\alpha+\beta,\ \alpha-\beta \\ \gamma\end{matrix} ; x\right]$

(b) If $f(x, y) = (x+y)^n$, n being a positive integer.,

$$= \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r,$$

then

$$\varphi(p, q) = \sum_{r=0}^{n} \left[\frac{n(n-1)\dots(n-r+1)}{r!} \frac{\Gamma_*(n-r-k+1\pm m)}{\Gamma(n-r-2k+1)} \frac{\Gamma_*(r-k_1+1\pm m_1)}{\Gamma(r-2k_1+1)} p^{-n+r} q^{-r} \right],$$

$R(-k+1\pm m) > 0$, $R(-k_1+1\pm m_1) > 0$ and $R(p, q) > 0$.

Further, if $k = \pm m$ and $k_1 = \pm m_1$, we get [McLachlan and Humbert, 1941, p. 62]

$$(x+y)^n \doteqdot \frac{n!}{(pq)^n} \frac{q^{n+1} - p^{n+1}}{q-p}$$

(c) If $f(x, y) = (xy)^{\frac{1}{2}n} J_n(2(xy)^{\frac{1}{2}})$,

then

$$\varphi(p, q) = pq \int_0^\infty \int e^{\frac{1}{2}px - \frac{1}{2}qy} W_{k+\frac{1}{2}, m}(px) W_{k_1+\frac{1}{2}, m_1}(qy) (px)^{-k-\frac{1}{2}} (qy)^{-k_1-\frac{1}{2}} \times$$
$$(xy)^{\frac{1}{2}n} J_n(2(xy)^{\frac{1}{2}}) dx dy$$

Now evaluating the x-integral by the relation (7) [Jaiswal, 1952, p. 389] we obtain

$$\varphi(p, q) = \frac{\Gamma_*(n-k+1\pm m) p^{-n} q}{\Gamma(n+1)\Gamma(n-2k+1)} \int_0^\infty y^n e^{-\frac{1}{2}qy} W_{k_1+\frac{1}{2}, m_1}(qy) (qy)^{-k_1-\frac{1}{2}} \times$$
$${}_2F_2\left[\begin{matrix} n-k+1\pm m \\ n+1, n-2k+1 \end{matrix} ; \frac{-y}{p} \right] dy$$

Now expanding ${}_2F_2$ into infinite series and changing the order of integration and summation, which can easily be justified, and using the Goldstein's integral [Goldstein, 1932, p. 114]

$$(A) \qquad \int_0^\infty x^{l-1} e^{-(\alpha^2+\frac{1}{2})x} W_{k, m}(x) dx = \frac{\Gamma_*(l+\frac{1}{2}\pm m)}{\Gamma(l+1-k)} {}_2F_1\left[\begin{matrix} l+\frac{1}{2}\pm m \\ l-k+1 \end{matrix} ; -\alpha^2 \right]$$

$R(l+\frac{1}{2}\pm m) > 0$, $R(\alpha^2+1) > 0$,

We have (with $\alpha^2 = 0$)

$$\varphi(pq) = (pq)^{-n} \sum_{r=0}^{\infty} \frac{(-)^r}{r!} \frac{1}{(pq)^r} \frac{\Gamma_*(n-k+1\pm m+r)\,\Gamma_*(n-k_1+1\pm m_1+r)}{\Gamma(n+1+r)\Gamma(n-2k+1+r)\Gamma(n-2k_1+1+r)}$$

$$= \frac{(pq)^{-n}\Gamma_*(n-k+1\pm m)\,\Gamma_*(n-k_1+1\pm m_1)}{n!\,\Gamma(n-2k+1)\,\Gamma(n-2k_1+1)} \times$$

$$\times {}_4F_3\left[\begin{matrix} n-k+1\pm m, n-k_1+1\pm m_1 \\ n+1, n-2k+1, n-2k_1+1 \end{matrix} ; \frac{-1}{pq} \right]$$

$R(n-k+1\pm m) > 0$, $R(n-k_1+1\pm m_1) > 0$, $R(p, q) > 0$ and $|pq| > 1$.

If $k = \pm m$ and $k_1 = \pm m_1$, we have [Bose, 1952, p. 27]

$$(xy)^{\frac{1}{2}}J_n(2(xy)^{\frac{1}{2}}) \doteqdot \frac{\Gamma(n+1)pq}{(pq+1)^{n+1}}.$$

(*d*) If in (*c*) we take $n = \frac{1}{2}$ and using the well known result

$$(xy)^{\frac{1}{2}n}J_{\frac{1}{2}}(2(xy)^{\frac{1}{2}}) = \frac{1}{\pi^{\frac{1}{2}}}\sin(2(xy)^{\frac{1}{2}}),$$

we have

$$\sin(2(xy)^{\frac{1}{2}})\,\frac{k+\frac{1}{2},\ k_1+\frac{1}{2}}{m,\ m_1} \rightarrow \frac{2}{(pq)^{\frac{1}{2}}}\,\frac{\Gamma_*(\frac{3}{2}-k\pm m)\,\Gamma_*(\frac{3}{3}-k_1\pm m_1)}{\Gamma(\frac{3}{2}-2k)\,\Gamma(\frac{3}{2}-2k_1)} \times$$

$$\times\, {}_4F_3\left[\begin{matrix}\frac{3}{2}-k\pm m,\ \frac{3}{2}-k_1\pm m_1 \\ \frac{3}{2},\ \frac{3}{2}-2k,\ \frac{3}{2}-2k_1\end{matrix};\ \frac{-1}{pq}\right]$$

If $k = \pm m$ and $k_1 = \pm m_1$, we obtain [Bose, 1952, p. 27]

$$\sin(2(xy)^{\frac{1}{2}}) \doteqdot \frac{\pi pq}{2(pq+1)^{3/2}}.$$

(*e*) If in (*c* we take $n = -\frac{1}{2}$ and using the well known result

$$(xy)^{\frac{1}{2}}J_{-\frac{1}{2}}(2(xy)^{\frac{1}{2}}) = \frac{1}{\pi^{\frac{1}{2}}}\cos(2(xy)^{\frac{1}{2}}),$$

we get [Bose, 1952, p. 28]

$$\frac{1}{(xy)^{\frac{1}{2}}}\cos(2(xy)^{\frac{1}{2}}) \doteqdot \frac{\pi pq}{(pq+1)^{\frac{1}{2}}}$$

(*f*) If
$$f(x, y) = x^{\nu}y^{\nu_1}e^{-ax-a_1y},$$

then

$$\varphi(p, q) = \frac{1}{p^{\nu}q^{\nu_1}}\ \frac{\Gamma_*(\nu-k+1\pm m)\,\Gamma_*(\nu_1-k_1+1\pm m_1)}{\Gamma(\nu-2k+1)\ \Gamma(\nu_1-2k_1+1)} \times$$

$$\times\, {}_2F_1\left[\begin{matrix}\nu-k+1\pm m \\ \nu-2k+1\end{matrix};\ \frac{-a}{p}\right]{}_2F_1\left[\begin{matrix}\nu_1-k_1+1\pm m_1 \\ \nu_1-2k_1+1\end{matrix};\ \frac{-a_1}{q}\right]$$

$R(\nu-k+1\pm m) > 0$, $R(\nu_1-k_1+1\pm m_1) > 0$, $R(p+a) > 0$, $R(q+a_1) > 0$, $|p| \geq |a$ and $|q| > |a_1|$.

If we put $k = \pm m$ and $k_1 = \pm m_1$, we obtain

$$x^{\nu}y^{\nu_1}e^{-ax-a_1y} \doteqdot \frac{\Gamma(\nu+1)\,\Gamma(\nu_1+1)}{(a+p)^{\nu+1}(a_1+q)^{\nu_1+1}}\,pq,$$

$R(\nu+1) > 0$, $R(\nu_1+1) > 0$, $R(p+a) > 0$ and $R(q+a_1) > 0$.

(*g*) If $f(x, y) = H_\nu(2(xy)^{\frac{1}{2}})$,

where $H_\nu(z)$ is Struve's function, then

$$\varphi(p, q) = \frac{2}{\pi^{\frac{1}{2}}(pq)^{\frac{1}{2}(\nu+1)}} \frac{\Gamma_*(\frac{1}{2}\nu - k + \frac{3}{2} \pm m)\,\Gamma_*(\frac{1}{2}\nu - k_1 + \frac{3}{2} \pm m_1)}{\Gamma(\nu + \frac{3}{2})\Gamma(\nu - 2k + \frac{3}{2})\Gamma(\frac{1}{2}\nu - 2k_1 + \frac{3}{2})} \times$$

$$\times {}_5F_4\left[\begin{matrix} 1, \frac{1}{2}\nu - k + \frac{3}{2} \pm m, \frac{1}{2}\nu - k_1 + \frac{3}{2} \pm m_1 \\ \frac{3}{2}, \nu + \frac{3}{2}, \frac{1}{2}\nu - 2k + \frac{3}{2}, \nu - 2k_1 + \frac{3}{2} \end{matrix} ; \frac{-1}{pq}\right]$$

$R(\frac{1}{2}\nu - k + \frac{3}{2} \pm m) > 0$, $R(\frac{1}{2}\nu - k_1 + \frac{3}{2} \pm m_1) > 0$, $R(p, q) > 0$ and $|pq| > 1$.

Further, if $k = \pm m$ and $k_1 = \pm m_1$, we get [Bose, 1952, p. 29]

$$H_\nu(2(xy)^{\frac{1}{2}} \doteqdot \frac{2\{\Gamma(\frac{1}{2}\nu + \frac{3}{2})\}^2}{(pq)^{\frac{1}{2}\nu + \frac{1}{2}}\pi^{\frac{1}{2}}\Gamma(\nu + \frac{3}{2})} {}_3F_2\left[\begin{matrix} 1, \frac{1}{2}\nu + \frac{3}{2}, \frac{1}{2}\nu + \frac{3}{2} \\ \frac{3}{2}, \nu + \frac{3}{2} \end{matrix} ; \frac{-1}{pq}\right],$$

$R(\frac{1}{2}\nu + \frac{3}{2}) > 0$, $R(p, q) > 0$ and $|pq| > 1$.

(*h*) If $f(x, y) = L_\nu(2(xy)^{\frac{1}{2}})$,

where $L_\nu(2)$ is Struve's function for imaginary argument, then

$$\varphi(p, q) = \frac{2}{\pi^{\frac{1}{2}}(pq)^{\frac{1}{2}(\nu+1)}} \frac{\Gamma_*(\frac{1}{2}\nu - k + \frac{3}{2} \pm m)\,\Gamma_*(\frac{1}{2}\nu - k_1 + \frac{3}{2} \pm m_1)}{\Gamma(\nu + \frac{3}{2})\Gamma(\frac{1}{2}\nu - 2k + \frac{3}{2})\Gamma(\frac{1}{2}\nu - 2k_1 + \frac{1}{2})} \times$$

$$\times {}_5F_4\left[\begin{matrix} 1, \frac{1}{2}\nu - k + \frac{3}{2} \pm m, \frac{1}{2}\nu - k_1 + \frac{3}{2} \pm m_1 \\ \frac{3}{2}, \nu + \frac{3}{2}, \frac{1}{2}\nu - 2k + \frac{3}{2}, \frac{1}{2}\nu - 2k_1 + \frac{3}{2} \end{matrix} ; \frac{1}{pq}\right],$$

$R(\frac{1}{2}\nu - k + \frac{3}{2} \pm m) > 0$, $R(\frac{1}{2}\nu - k_1 + \frac{3}{2} \pm m_1 > 0$ $R(p, q) > 0$ and $|pq| > 1$.

Further, if we take $k = \pm m$ and $k_1 = \pm m_1$, we get [Bose, 1952, p. 29]

$$L_\nu(2(xy)^{\frac{1}{2}}) \doteqdot \frac{2\{\Gamma(\frac{1}{2}\nu + \frac{3}{2})\}^2}{\pi^{\frac{1}{2}}(pq)^{\frac{1}{2}\nu + \frac{1}{2}}\,\Gamma(\nu + \frac{3}{2})} {}_3F_2\left[\begin{matrix} \frac{1}{2}\nu + \frac{3}{2}, \frac{1}{2}\nu + \frac{3}{2}, 1 \\ \frac{3}{2}, \nu + \frac{3}{2} \end{matrix} ; \frac{1}{pq}\right],$$

$R(\frac{1}{2}\nu + \frac{3}{2}) > 0$, $R(p, q) > 0$ and $|pq| > 1$.

(*i*) If $f(x, y) = I_\nu(2(xy)^{\frac{1}{2}})$,

where $I_\nu(z)$ is Bessel's function for imaginary argument. Then

$$\varphi(p, q) = \frac{1}{(pp)^{\frac{1}{2}\nu}} \frac{\Gamma_*(\frac{1}{2}\nu - k + 1 \pm m)\Gamma_*(\frac{1}{2}\nu - k_1 + 1 \pm m_1)}{\Gamma(\nu + 1)\Gamma(\frac{1}{2}\nu - 2k + 1)\Gamma(\frac{1}{2}\nu - 2k_1 + 1)} \times$$

$$\times {}_4F_3\left[\begin{matrix} \frac{1}{2}\nu - k + 1 \pm m, \frac{1}{2}\nu - k_1 + 1 \pm m_1 \\ \nu + 1, \frac{1}{2}\nu - 2k + 1, \frac{1}{2}\nu - 2k_1 + 1 \end{matrix} ; \frac{1}{pq}\right]$$

$R(\frac{1}{2}\nu - k + 1 \pm m) > 0$, $R(\frac{1}{2}\nu - k_1 + 1 \pm m_1) > 0$, $R(p, q) > 0$ and $|pq| > 1$.

Further, if $k = \pm m$ and $k_1 = \pm m_1$, we get [Bose, 1952, p. 30].

$$I_\nu(2(xy)^{\frac{1}{2}}) \doteqdot \frac{1}{(pq)^{\frac{1}{2}\nu}} \frac{\{\Gamma(\frac{1}{2}\nu + 1)\}^2}{\Gamma(\nu + 1)} {}_2F_1\left[\begin{matrix} \frac{1}{2}\nu + 1, \frac{1}{2}\nu + 1 \\ \nu + 1 \end{matrix} ; \frac{1}{pq}\right],$$

$R(\nu) > -1$.

(*j*) If* $$f(x, y) = X^b_m(2(2xy)^{\frac{1}{2}})$$

$$= \mathrm{ber}^2_n\{(2xy)^{\frac{1}{2}}\} + \mathrm{bei}^2_n\{2(2xy)^{\frac{1}{2}}\},$$

where $$X^b_n\{2(2xy)^{\frac{1}{2}}\} = \frac{2^n x^n y^n}{n\,!\,n\,!}\,{}_0F_3[n+1, \tfrac{1}{2}n+1, \tfrac{1}{2}n+\tfrac{1}{2}\,;\,x^2y^2]$$

and ber(x) and ber(x) are Kelvin's functions.

Therefore,

$$\varphi(p,q) = \frac{1}{(pq)^n}\,\frac{2^n}{(n\,!)^2\Gamma(n+1)\Gamma(\frac{1}{2}n+1)\Gamma(\frac{1}{2}n+\frac{1}{2})}\sum_{r=0}^{\infty}\Bigg[\frac{1}{r\,!\,\Gamma(n+r+1)\Gamma(\frac{1}{2}n+r+1)\Gamma(\frac{1}{2}n+r+\frac{1}{2})(pq)2r}$$

$$\times\frac{\Gamma_*(n-k+2r+1\pm m)\Gamma_*(n-k_1+2r+1\pm m_1)}{\Gamma(n-2k+2r+1)\Gamma(n-2k_1+2r+1)}\Bigg]$$

$$= \frac{2^n}{(pq)^n}\,\frac{\Gamma_*(n-k+1\pm m)\Gamma(n-k_1+1\pm m_1)}{\Gamma(n-2k+1)\Gamma(n-2k_1+1)\{n\,!\,\Gamma(n+1)\Gamma(\frac{1}{2}n+1)\Gamma(\frac{1}{2}n+\frac{1}{2})\}^2}$$

$$[\times\,{}_8F_7\left\{\begin{matrix}\frac{n-k+1\pm m}{2}, \frac{n-k+2\pm m}{2}, \frac{n-k_1+1\pm m_1}{2}, \frac{n-k_1+2\pm m_1}{2} & \\ & ;\ \frac{16}{p^2q^2} \\ n+1, \frac{n}{2}+1, \frac{n}{2}+\frac{1}{2}, \frac{n-2k+1}{2}, \frac{n-2k+2}{2}, \frac{n-2k_1+1}{2}, \frac{n-2k_1+2}{2} & \end{matrix}\right\},$$

$$R(n-k+1\pm m) > 0,\ R(n-k_1+1\pm m_1) > 0,\ R(p, q) > 0 \text{ and } |pq| > 4.$$

If $k = \pm m$ and $k_1 = \pm m_1$, we get [Bose, 1952, p. 31]

$$X^b_n(2(2xy)^{\frac{1}{2}}) \doteqdot \frac{2^n}{(pq)^n}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}n+\frac{1}{2}, \frac{1}{2}n+1 \\ n+1\end{matrix}\ ;\ \frac{16}{p^2q^2}\right].$$

III

We shall now obtain some properties of the Meijer transform by utilising the properties of Generalised Laguerre polynomial, Parabolic Cylinder and Batemen's k-functions.

Theorem I. *If* $\varphi_{n-r,n-r}(p, q)$ *is* $L^{\alpha+2r}_{n-r}L^{\alpha+2r}_{n-r}-$ *transorm of* $q(x, y)$, *then*

$$pq\int_0^{\infty}\int(pqxy)^{-n}e^{-xp-qy}L^{\alpha}_n(px+qy)f(x, y)\,dx\,dy$$

$$= \sum_{r=0}^{n}\frac{(-)^r(\alpha+2r)\Gamma(\alpha+2r)}{(n-r)\,!\,r\,!\,\Gamma(\alpha+n+r+1)}\,\varphi_{n-r,\,n-r}(p, q).$$

the integrals involved are absolutely convergent.

---

[*] It is easy to see, from the asymptotic expansions of ber(z) and bei(z) that
$$X^b_n\{(2(2x)^{\frac{1}{2}})\} = O\ \{x^{-\frac{1}{2}}\exp(4(x)^{\frac{1}{4}})\}$$

Proof. We have

$$\varphi_{n-r,\,n-r}(p,q) = \{(n-r)!\}^2 pq \int_0^\infty\!\!\int (pqxy)^{-n+r} e^{-xp-qy} L_{n-r}^{\alpha+2r}(px) L_{n-r}^{\alpha+2r}(qy) f(xy)\,dx\,dy$$

Multiplying both sides of this equation by $\dfrac{(-)^r(\alpha+2r)\Gamma(\alpha+r)}{(n-r)!\,r!\,\Gamma(\alpha+n+r+1}$,

and taking the sum from $r=0$ to $r=n$, we get

$$\sum_{r=0}^{n} \frac{(+)^r(\alpha+2r)\Gamma(\alpha+r)}{(n-r)!\,r!\,\Gamma(\alpha+n+r+1)}\varphi_{n-r,\,n-r}(p,q)$$

$$= pq\int_0^\infty\!\!\int (pqxy)^{-n} e^{-xp-qy} \sum_{r=0}^{n} \frac{(-)^r(\alpha+2r)\Gamma(\alpha+r)(n-r)!}{r!\,\Gamma(\alpha+n+r+1)} L_{n-r}^{\alpha+2r}(px) L_{n-r}^{\alpha+2r}(qy)$$

$$\times (pq\,xy)^r f(x,y)\,dx\,dy$$

Now using the result [Burchanal and Chaundy, 1941, p. 127]

$$(B)\; L_n^\alpha(x+y) = \sum_{r=0}^{n}\left[\frac{(-)^r(n-r)!\,(\alpha+2r)\Gamma(\alpha+r)}{r!\,\Gamma(\alpha+n+r+1)}\right.(xy)^r L_{n-r}^{\alpha+2r}(x) L_{n-r}^{\alpha+2r}(y)$$

we have

$$\sum_{r=0}^{n} \frac{(-)^r(\alpha+2r)\,!\,\Gamma(\alpha+r)}{(n-r)!\,r!\,\Gamma(\alpha+n+r+1)}\varphi_{n-r,\,n-r}(p,q)$$

$$= pq\int_0^\infty\!\!\int (pqxy)^{-n} e^{-xp-qy} L_n^\alpha(px+qy) f(x,y)\,dx\,dy,$$

provided the integrals are absolutely convergent.

Example. If $f(x,y) = (xy)^{\frac{1}{2}n} J_n(2(xy)^{\frac{1}{2}})$. then using Example (*c*) with $k = \frac{1}{2}\alpha+n$ and $m = \frac{1}{2}\alpha+r$, we have

$$\varphi_{n-r,\,n-r}(p,q) = \frac{(pq)^{-n}}{n!}\left[\frac{\Gamma(2r+1)\Gamma(1-\alpha)}{\Gamma(1-\alpha-n)}\right]^2 {}_1F_3\left[\begin{matrix}2r+1,\,2r+1,\,1-\alpha,\,1-\alpha\\ n+1,\,1-\alpha-n,\,1-\alpha-n\end{matrix}\;;\;\frac{-1}{pq}\right] \quad (8.1)$$

Hence using Theorem I we have

$$(pq)^{1-n}\int_0^\infty\!\!\int (xy)^{-n+\frac{1}{2}} e^{-xp-qy} L_n^\alpha(px+qy) J_n(2(xy)^{\frac{1}{2}})\,dx\,dy$$

$$= \sum_{r=0}^{n} \frac{(-)^r(\alpha+2r)\Gamma(\alpha+r)}{(n-r)!\,r!\,\Gamma(\alpha+n+r+1)}\varphi_{n-r,\,n-r}(p,q),$$

where $\varphi_{n-r,\,n-r}(p,q)$ is given by (8.1).

Theorem II. *If* $\varphi_{n,n}(p,q) \underset{\pm\frac{1}{2},\,+\frac{1}{2}}{\overset{\frac{1}{2}n+\frac{1}{2},\,\frac{1}{2}n+\frac{1}{2}}{\doteqdot}} x^{\frac{1}{2}n} y^{\frac{1}{2}n} f(x,y)$, *where* $f(x,y)$ *is not a function of* n; *then*

$$\sum_{n=0}^{\infty} \frac{(2t)^n(pq)^{\frac{1}{2}n}}{n!}\varphi_{n,n}(p,q) = \frac{pq}{(1-t^2)^{\frac{1}{2}}}\int_0^\infty\!\!\int exp\left[\frac{2t(pqxy)^{\frac{1}{2}}-(px+qy)}{1-t^2}\right] f(x,y)\,dx\,dy \quad (8.2)$$

provided $R(\mu_1+1) > 0$, $R(\mu_2+1) > 0$ *where* $f(x, y) = O(x^{\mu_1})$ *for small* x *and* $f(x, y) = O(y^{\mu_2})$ *for small* y, $f(x, y)$ *is a continuous function of* x *and* y *for* $x > 0$ *and* $y > 0$, $|t| < 1$ *and the integral in (3.2) is absolutely convergent.*

Proof. We have

$$2^n(pq)^{\frac{1}{2}n}\varphi_{n,n}(p, q) = pq\int_0^\infty\int e^{-\frac{1}{2}(px+qy)}D_n(2px)^{\frac{1}{2}}D_n(2qy)^{\frac{1}{2}}f(x, y)dxy.$$

Multiplying both sides of this equation by $\frac{t^n}{n!}$ and taking the sum from $n = 0$ to $n = \infty$ and changing the order of integration and summation, we have

$$\sum_{n=0}^{\infty}\frac{(2t)^n(pq)^{\frac{1}{2}n}}{n!}\varphi_{n,n}(p, q)$$

$$= pq\int_0^\infty\int e^{-\frac{1}{2}(px+qy)}\sum_{n=0}^{\infty}\frac{t^n}{n!}D_n(2px)^{\frac{1}{2}}D_n(2qy)^{\frac{1}{2}}f(x, y)dxdy \qquad (3.2.1)$$

On using the Mehler's formula [Watson, 1933, p. 195]

$$(C)\quad \sum_{n=0}^{\infty}\frac{t^n}{n!(\pi/2)^{\frac{1}{2}}}D_n(x)D_n(y) = \frac{1}{(\pi/2)^{\frac{1}{2}}(1-t^2)}\exp\left\{\frac{4xyt-(x^2+y^2)(1+t^2)}{4(1-t^2)}\right\}, |t| < 1,$$

we have

$$\sum_{n=0}^{\infty}\frac{2^nt^n(pq)^{\frac{1}{2}n}}{n!}\varphi_{n,n}(p, q) = \frac{pq}{(1-t^2)^{\frac{1}{2}}}\int_0^\infty\int\exp\left\{\frac{2t(pqxy)^{\frac{1}{2}}-(px+qy)}{1-t^2}\right\}f(x, y)dxdy \qquad (3.2)$$

Regarding the change of order of integration and summation in (3.2.1) the series is uniformly and absolutely convergent for $|t| < 1$ and the integral is uniformly and absolutely convergent for $x \geq 0$ and $y \geq 0$, if $R(\mu_1+1) > 0$ and $R(\mu_2+1) > 0$, where $f(x, y) = O(x^{\mu_1})$ for small x and $f(x, y) = O(y^{\mu_2})$ for small y, $R(p) \geq p_0 > 0$ and $R(q) \geq q_0 > 0$; $f(x, y)$ is a continuous function of x and y for $x > 0$ and $y > 0$. Also the integral in (3.2) is absolutely convergent. Hence the change of order is justified with the above conditions.

Example. Let $f(x, y) = x^\nu y^{\nu+\frac{1}{2}}$.

Then, from example (a),

$$\varphi_{n,n}(p, q) = \frac{\pi\Gamma(2\nu+2)\Gamma(2\nu+3)p^{-\frac{1}{2}n-\nu}q^{-\frac{1}{2}n-\nu-\frac{1}{2}}}{2^{4\nu+8}\Gamma(\frac{3}{2}+\nu-\frac{1}{2}n)\Gamma(2+\nu-\frac{1}{2}n)}$$

Hence from theorem II, we get

$$\frac{1}{(1-t^2)^{\frac{1}{2}}}\int_0^\infty\int\exp\left\{\frac{2(pqxy)^{\frac{1}{2}}t-(px+qy)}{1-t^2}\right\}x^\nu y^{\nu+\frac{1}{2}}dxdy$$

$$= \frac{\pi\Gamma(2\nu+2)\Gamma(2\nu+3)}{2^{4\nu+8}(pq)^{\nu+1}q^{\frac{1}{2}}}\sum_{n=0}^{\infty}\frac{2^nt^n}{n!\,\Gamma(\nu+\frac{3}{2}-\frac{1}{2}n)\,\Gamma(\nu+2-\frac{1}{2}n)} = \frac{\pi^{\frac{1}{2}}\Gamma(2\nu+2)(1+t)^{2\nu+2}}{p^{\nu+1}q^{\nu+3/2}2^{2\nu+1}},$$

$$R(\nu+\tfrac{3}{2}) > 0,\ R(p, q) > 0 \text{ and } |t| < 1$$

Theorem III. *If* $x^n y^n f(x, y) \xrightarrow[\frac{1}{2},\frac{1}{2}]{n,\, n} \varphi_{n-\frac{1}{2},\, n-\frac{1}{2}}(p, q)$, *where* $f(x, y)$ *is not a function of* n, *then, provided the integral and the series conveiges and* $|t| < 1$,

$$\sum_{n=1}^{\infty} \frac{np^n q^n t^{n-1}}{(n\,!)^2} \varphi_{n-\frac{1}{2},\, n-\frac{1}{2}}(p, q) = \frac{t^{-\frac{1}{2}}}{1-t}(pq)^{3/2} \int_0^{\infty}\!\!\int (xy)^{\frac{1}{2}} \exp\left\{\frac{-(px+qy)}{1-t}\right\} I_1\left\{\frac{2(pqxyt)^{\frac{1}{2}}}{1-t}\right\} f(x, y)\,dxdy$$

Proof. We have

$$\varphi_{n-\frac{1}{2},\, n-\frac{1}{2}}(p, q) = pq \int_0^{\infty}\!\!\int e^{-\frac{1}{2}(px+qy)} W_{n,\frac{1}{2}}(px) W_{n,\frac{1}{2}}(qy) . (px)^{-n} (qy)^{-n} x^n y^n f(x, y)\,dxdy$$

$$= (n\,!)^2 (pq)^{1-n} \int_0^{\infty}\!\!\int e^{-\frac{1}{2}(px+qy)} k_{2n}(\tfrac{1}{2}px) k_{2n}(\tfrac{1}{2}qy) f(x, y\; dxdy,$$

on using the well-known result

$W_{\nu,\frac{1}{2}}(t) = \Gamma(\nu+1) k_{2\nu}(\frac{1}{2}t)$, where $k_n(s)$ is the Bateman's function.

Multiplying both sides of this equation by nt^{n-1} and taking the sum from $n = 1$ to $n = \infty$, changing the order of summation and integration and using the result due to Shabde [Shabde, 1936-37, p. 277]

$$(D) \quad \sum_{n=1}^{\infty} nt^{n-1} k_{2n}(x) k_{2n}(y) = 2(xy)^{\frac{1}{2}} \frac{t^{-\frac{1}{2}}}{1-t} \exp\left\{-(x+y)\frac{1+t}{1-t}\right\} I_1\left\{\frac{4(xyt)^{\frac{1}{2}}}{1-t}\right\}, \; |t| < 1,$$

we obtain $$\sum_{n=1}^{\infty} \frac{p^n q^n t^{n-1}}{n\,!(n-1)\,!} \varphi_{n-\frac{1}{2},\, n-\frac{1}{2}}(p, q)$$

$$= \frac{t^{-\frac{1}{2}}}{1-t}(pq)^{3/2} \int_0^{\infty}\!\!\int (xy)^{\frac{1}{2}} \exp\left\{\frac{-(px+qy)}{1-t}\right\} I_1\left\{\frac{2(pqxyt)^{\frac{1}{2}}}{1-t}\right\} f(x, y)\,dxdy.$$

Theorem IV. *If*

$$(st)^{-n+\frac{1}{2}} f(s, t) \xrightarrow[m,\, m_1]{k+\frac{1}{2},\, k_1+\frac{1}{2}} \varphi_n(p, q)$$

and $$\psi(x, y) \doteqdot e^{-\frac{1}{2}(sp+tq)} s^{-(n+k)} t^{-(n+k_1)} W_{k+\frac{1}{2}, m}(ps) W_{k_1+\frac{1}{2}, m_1}(qt) f(s, t),$$

then $$\varphi_{n+l}(p, q) = \frac{p^{-k+\frac{1}{2}} q^{-k_1+\frac{1}{2}}}{[\Gamma(l)]^2} \int_0^{\infty}\!\!\int \psi(x, y)(xy)^{l-2}\,dxdy,$$

provided the integrals converge absolutely and the Meijer transform of $|(st)^{-n+\frac{1}{2}} f(s, t)|$ *exists.*

Proof. We have from definitions of Meijer transform of two variables and Laplace transform that

$$\varphi_{n+l}(p, q) = p^{-k+\frac{1}{2}} q^{-k_1+\frac{1}{2}} \int_0^{\infty}\!\!\int e^{-\frac{1}{2}ps} W_{k+\frac{1}{2}, m}(ps)\, s^{-l-n-k} e^{-\frac{1}{2}qt} W_{k_1+\frac{1}{2}, m}(qt)\, t^{-l-n-k_1} f(s, t)\,dsdt \tag{4.10}$$

and

$$\psi(x, y) = xy\int_0^\infty\int e^{-xs-ty}e^{-\frac{1}{2}(sp+tq)}s^{-(n+k)}t^{-(n+k_1)}W_{k+\frac{1}{2}, m}(ps)W_{k_1+\frac{1}{2}, m_1}(qt)f(s, t)\,ds\,dt \qquad (4.11)$$

Also

$$s^{-l} = \frac{1}{\Gamma(l)}\int_0^\infty e^{-sx}x^{l-1}dx$$

and

$$t^{-l} = \frac{1}{\Gamma(l)}\int_0^\infty e^{-ty}y^{l-1}dx.$$

Substituting these values of s^{-l} and t^{-l} in (4.1) changing the order of integration and evaluating the integral with respect to s and t by the help of (4.11), we get

$$\varphi_{n+l}(p, q) = \frac{p^{-k+\frac{1}{2}}q^{-k_1+\frac{1}{2}}}{[\Gamma(l)]^2}\int_0^\infty\int (xy)^{l-2}\psi(x, y)\,dx\,dy.$$

The change of order of integration may be justified by de la Vallée Poussin's Theorem. [Bromwich, 1.47, p. 504]

Example. Let $f(s, t) = s^{\mu}t^{\mu_1}$.

Then from example (*a*)

$$\varphi_{n+l}(p, q) = \frac{\Gamma_*(\mu+3/2-n-l-k\pm m)\Gamma_*(\mu_1+3/2-n-l-k_1\pm m_1)}{\Gamma(\mu+3/2-n-l-2k)\Gamma(\mu_1+3/2-n-l-2k_1)}\,p^{-\mu-\frac{1}{2}+n+l}q^{-\mu_1-\frac{1}{2}+n} \qquad (4.12)$$

$$R(\mu+3/2-n-l-k\pm m)>0, \quad R(\mu_1+3/2-n-l-k_1\pm m_1)>0 \text{ and } R(p, q)\geq 0$$

and

$$\psi(x, y) = \int_0^\infty\int e^{-sx-ty}e^{-\frac{1}{2}(sp+tq)}s^{-(n+k-\mu)}t^{-(n+k_1-\mu)}W_{k+\frac{1}{2}, m}(ps)W_{k_1+\frac{1}{2}, m_1}(qt)\,ds\,dt$$

Using example (*f*) we get

$$\psi(x, y) = p^{n-\mu+k-1}\frac{\Gamma_*(-n+\mu-k\pm m+3/2)}{\Gamma(-n+\mu-2k+3/2)}\,{}_2F_1\left[\begin{matrix}\mu-k-n+3/2\pm m\\ \mu-n-2k+3/2\end{matrix};\frac{-x}{p}\right]$$

$$\times q^{-n-\mu_1+k_1-1}\frac{\Gamma_*(-n+\mu_1-k_1\pm m_1+3/2}{\Gamma(-n+\mu_1-2k_1+3/2)}\,{}_2F_1\left[\begin{matrix}\mu_1-k_1-n+3/2\pm m_1\\ \mu_1-n-2k_1+3/2\end{matrix};\frac{-y}{q}\right].$$

Hence by Theorem IV we have

$$\frac{p^{n-\mu-1}q^{n-\mu_1-1}}{[\Gamma(l)]^2}\frac{\Gamma_*(-n+\mu-k+3/2\pm m)\Gamma_*(-n+\mu_1-k_1\pm m_2+3/2)}{\Gamma(-n+\mu-2k+3/2)\Gamma(-n+\mu_1-2k_1+3/2)}$$

$$\times\int_0^\infty\int (xy^{\,l-2}\,{}_2F_1\left[\begin{matrix}\mu-k-n+3/2\pm m\\ \mu-n-2k+3/2\end{matrix};-\frac{x}{p}\right]{}_2F_1\left[\begin{matrix}\mu_1-k_1-n+3/2\pm m_1\\ \mu_1-n-2k_1+3/2\end{matrix};\frac{-y}{q}\right]\times$$

$$dx\,dy = \varphi_{n+l}(p, q).$$

where $\varphi_{n+l}(p, q)$ is given by (4.12) and

$R(\mu+3/2-n-l-k\pm m) > 0, \quad R(\mu_1+3/2-n-l\pm m_1) > 0, \quad R(l) \geq 1$ and $R(p.\, q) > 0$

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ON TWO THEOREMS OF OPERATIONAL CALCULUS

By

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1. The object of this paper is to obtain two theorems of operational calculus. The theorems are illustrated by examples which involve generalised K-function of Bateman, Sonine's polynomial and MacRobert's E-function. The theorems are believed to be new.

2. *If*

$$\varphi(p) \doteqdot x^{\nu}h(x)$$

$$h(p) \doteqdot f(x)$$

and

$$p^{2-\lambda}f(p \doteqdot g(x),$$

then (Bose 1952, p. 128)

$$\varphi(p) = \frac{\Gamma(\nu+2)\,\Gamma(\lambda)}{p^{\frac{1}{2}(\nu-\lambda+1)}} \int_0^\infty e^{\frac{1}{2}sp} s^{\frac{1}{2}(\nu-\lambda+1)} W_{-\frac{1}{2}(\lambda+\nu+1),\ \frac{1}{2}(\lambda-\nu-2)}(sp)g(s)ds \tag{2.1}$$

provided $R(p) > 0$, $R(\nu+2) > 0$, $R(\lambda) > 0$, λ not an integer, the Laplace transform of $|g(s)|$ exists and the integral in (2 1) is absolutely convergent.

Now we state the following theorem

Theorem I. *If*

$$\psi(p) \doteqdot \varphi(x)$$

$$\varphi(p) \doteqdot x^{\nu}h(x)$$

$$h(h) \doteqdot g(x)$$

and

$$p^{2-\lambda}f(p) \doteqdot g(x),$$

then

$$\psi(p) = p\,\frac{\Gamma(\nu+2)\Gamma(\lambda)\Gamma(\lambda-\nu)}{\Gamma(\lambda+2)} \int_0^\infty s^{\nu-\lambda}\,{}_2F_1\left[\begin{matrix}\lambda-\nu,\ 2\\ \lambda+2\end{matrix};\ 1-\frac{p}{s}\right]g(s)ds \tag{2.2}$$

provided the conditions in (2.1) *are satisfied,* $R(\lambda-\nu) > 0$, *and the integral in* (2.2) *is absolutely convergent.*

Proof. In

$$\psi(p) = p\int_0^\infty e^{-px}\varphi(x)dx, \quad R(p) > 0 \tag{2.3}$$

Substitute the value of $\varphi(x)$, as obtained from (2.1), which gives

$$\psi(p) = p\int_0^\infty e^{-px}\left[\frac{\Gamma(\nu+2)\Gamma(\lambda)}{x^{\frac{1}{2}(\nu-\lambda+1)}}\int_0^\infty e^{\frac{1}{2}sx} s^{\frac{1}{2}(\nu-\lambda+1)} W_{-\frac{1}{2}(\lambda+\nu+1),\ \frac{1}{2}(\lambda-\nu-2)}(sx)g(s)ds\right]dx$$

On changing the order of integration

$$\psi(p) = p\Gamma(\nu+2)\Gamma(\lambda)\int_0^\infty s^{\frac{1}{2}(\nu-\lambda+1)}g(s)\left[\int_0^\infty e^{-px}e^{\frac{1}{2}sx}x^{\frac{1}{2}(\lambda-\nu-1)}W_{-\frac{1}{2}(\lambda+\nu+1),\,\frac{1}{2}(\lambda-\nu-2)}(sx)\,dx\right]ds \tag{2.4}$$

Now on evaluating the second integral with the help of Goldstein's integral (1932)

$$\int_0^\infty e^{-(\alpha^2+\frac{1}{2})x}x^{l-1}W_{k,\,m}(x)\,dx = \frac{\Gamma(l-m+\frac{1}{2})\Gamma(l+m+\frac{1}{2})}{\Gamma(l-k+1)}\,{}_2F_1\left\{\begin{matrix}l-m+\frac{1}{2},\, l+m+\frac{1}{2} \\ l-k+1\end{matrix}\,;\, -\alpha^2\right\}$$

$$R(l\pm m+\tfrac{1}{2}) > 0, \quad R(\alpha^2+1) > 0 \tag{2.5}$$

we immediately obtain our theorem.

As regards the change in the order of integration we observe that, x integral is absolutely convergent if $R(\lambda-\nu) > 0$. The s-integral is absolutely convergent because integral in (2.1) is absolutely convergent. The repeated integral exists because of the absolute convergence of the integral in (2 2). Hence the change in the order of integration is justified because of de la Vallee Poussin's theorem.

3. *Example* 1. We have

$$g(x) = \frac{(1+x)^{\lambda-\alpha-1}}{\Gamma(\lambda-\alpha)} \doteqdot e^{p}p^{\alpha-\lambda+1} = f(p)p^{2-\lambda}$$

$R(p) > 0, \quad R(\lambda-\alpha) > 0.$

Also
$$f(x) = e^{-x}x^{\alpha-1} \doteqdot \Gamma(\alpha)\frac{p}{(1+p)^{\alpha}} = h(p)$$

$R(p) > 0, \quad R(\alpha) > 0.$

We have also the operational representation [Bose N. N., 1950]

$$h(x)x^{\beta-2} = \Gamma(\alpha)x^{\beta-1}(1+x)^{-\alpha} \doteqdot \frac{1}{p^{\beta-1}}E(\alpha,\beta :: p) = \varphi(p)$$

$$R(p) > 0, \quad R(\beta) > 0. \tag{3.1}$$

Hence from (2.1) we obtain

$$E(\alpha,\beta :: p) = \frac{\Gamma(\beta)\Gamma(\lambda)}{\Gamma(\lambda-\alpha)}\,p^{\frac{1}{2}(\beta+\lambda-1)}\int_0^\infty e^{\frac{1}{2}sp}s^{\frac{1}{2}(\beta-\lambda-1)}(1+s)^{\lambda-\alpha-1}W_{-\frac{1}{2}(\lambda+\beta-1),\,\frac{1}{2}(\lambda-\beta)}(sp)\,ds \tag{3.2}$$

$R(\alpha-\beta) > 0$, $R(p) > 0$, $R(\lambda-\alpha) > 0$, $R(\alpha) > 0$ $R(\beta) > 0$ and $R(\beta-\lambda+1) > 0$, and λ is not an integer.

And further since (Ramkumar, 1952)

$$\varphi(x) = x^{-\beta+1}E(\alpha,\beta :: x) \doteqdot \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+2)}\,p^{\beta-2}\,{}_2F_1[\alpha,\beta\,;\,\alpha+2\,;\,1-p] = \psi(p)$$

where $R(\alpha-\beta+2) > 0, \quad 0 < p < 2.$

hence from (2.2)

$$\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha-\beta+2)}{\Gamma(\alpha+2)}\,p^{\beta-2}\,{}_2F_1[\alpha,\beta\,;\alpha+2;1-p]=p\,\frac{\Gamma(\beta)\Gamma(\lambda)\Gamma(\lambda-\beta+2)}{\Gamma(\lambda+2)\Gamma(\lambda-\alpha)}\times$$

$$\int_0^\infty s^{\beta-\lambda-2}(1+s)^{\lambda-\alpha-1}\,{}_2F_1\left[\lambda-\beta+2,2;\,\lambda+2\,;\,1-\frac{p}{s}\right]ds \qquad (3.3)$$

where $0<p<2$, $R(\lambda-\alpha)>0$, $R(\alpha)>0$, $R(\beta)>0$, $R(\alpha-\beta)>0$, $R(\beta-\lambda+1)>0$ and λ is not an integer.

Incidentally we state here a property of MacRobert's E-function which is easily obtained by applying a theorem due to Bose (1953).

If $\qquad \varphi_r(p)\doteqdot x^r f(x)$

and $\qquad \varphi(p)\equiv\varphi_0(p)$

then $\qquad \dfrac{p}{p+q}\varphi(p+q)=\displaystyle\sum_{r=0}^{\infty}(-1)^r\frac{q^r}{r!}\varphi_r(p),$

provided $f(x)$ *is continuous for* $x>0$, *and* $R(\mu_1)>0$ *where* $f(x)=O(x^{\mu_1})$ *for small* x, *and* $e^{-(p+q)x}f(x)$ *tends towards zero as* x *tends towards infinity.*

If we write (3.1) as

$$\varphi_r(p)=\frac{1}{p^r}E(\alpha,\,r+1::p)\doteqdot\Gamma(\alpha)x^r(1+x)^{-\alpha}=x^r f(x)$$

Then we immediately obtain

$$\frac{p}{p+q}E(\alpha,\,1::p+q)=\sum_{r=0}^{\infty}(-1)^r\frac{q^r}{r!\,p^r}E(\alpha,\,r+1::p) \qquad (3.5)$$

Example 2. This is an illustration of Bose's result (2.1)

We have (Chakravarty N. K., 1952, p. 66)

$$g(x)=e^{-\frac12 x}K_{2n}^{2l}(\tfrac12 x)\doteqdot(-1)^{n-l-1}\frac{p^{n-l}}{(1+p)^{n+l+1}}=p^{2-\lambda}f(p)$$

We also obtain with the help of the integral (Whittaker and Watson, 1952, p. 340)

$$W_{k,\,m}(z)=\frac{e^{-\frac12 z}z^k}{\Gamma(\frac12-k+m)}\int_0^\infty t^{-k-\frac12+m}\left(1+\frac{t}{z}\right)^{k-\frac12+m}e^{-t}\,dt,$$

$R(k-\frac12-m)\leq 0.$

the operational representation

$$f(x)=(-1)^{n-l-1}\frac{x^{n-l+\lambda-2}}{(1+x)^{n+l+1}}\doteqdot(-1)^{n-l-1}\,\frac{\Gamma(n+\lambda-l-1)}{p^{\frac12\lambda-l-3/2}}\,e^{\frac12 p}\,W_{\frac12-n-\frac12\lambda,\,\frac12\lambda-l-\frac12}(p)=h(p),$$

$R(p)>0,\quad R(n+\lambda-l-1)\geq 0.$

And with the help of (2.5) we obtain

$$x^{\nu}h(x) = (-1)^{n-l-1}\Gamma(n+\lambda-l-1)x^{\nu+l-\frac{1}{2}\lambda+3/2}e^{\frac{1}{2}x}W_{\frac{1}{2}-n-\frac{1}{2}\lambda,\ \frac{1}{2}\lambda-l-1}(x) \doteqdot (-1)^{n-l-1}\Gamma(n+\lambda-l-1)$$

$$\times\ \frac{\Gamma(\nu-\frac{1}{2}\lambda+2)\Gamma(\nu+2l-3\lambda/2+4)}{\Gamma(\nu+l-\frac{1}{2}\lambda+n+3)}\ {}_2F_1\left[\begin{matrix}\nu-\frac{1}{2}\lambda+2,\ \nu+2l-\frac{3}{2}\lambda+4\\ \nu+l-\frac{1}{2}\lambda+n+3\end{matrix};1-p\right] = \varphi(p),$$

$R(p) > 0$, $R(\nu-\lambda/2+2) > 0$ and $R(\nu+2l-\frac{3}{2}\lambda+4) > 0$.

Hence from (2.1) we have

$$\frac{\Gamma(n+\lambda-l-1)\Gamma(\nu-\frac{1}{2}\lambda+2)\Gamma(\nu+2l-3\lambda/2+4)}{\Gamma(\nu+l-\frac{1}{2}\lambda+n+3)}\ {}_2F_1\left[\begin{matrix}\nu-\frac{1}{2}\lambda+2,\ \nu+2l-3\lambda/2+4\\ \nu+l-\frac{1}{2}\lambda+n+3\end{matrix};\ 1-p\right]$$

$$= \frac{\Gamma(\nu+2)\Gamma(\lambda)}{p^{\frac{1}{2}(\nu-\lambda+1)}}(-1)^{n-l-1}\int_0^\infty e^{\frac{1}{2}sp}s^{\frac{1}{2}(\nu-\lambda+1)}W_{-\frac{1}{2}(\nu+\lambda+1),\ \frac{1}{2}(\lambda-\nu-2)}(sp)e^{-\frac{1}{2}s}K_{2n}^{2l}(\tfrac{1}{2}s)ds \tag{3.6}$$

$R(p) > 0$, $R(\nu+2l-\lambda+4) > 0$, $l > -1$, $R(\nu+2) > 0$, $R(\lambda) > 0$ and λ is not an integer.

4. Theorem II. *If*

$$\psi(p) \doteqdot f(x)$$

$$f(p) \doteqdot h(x)$$

$$p^{-2\nu}h(1/p) \doteqdot g(x)$$

and

$$F(p) \doteqdot x^{\frac{1}{2}-\nu-l}g(x),$$

then

$$\psi(p) = \frac{\Gamma(2\nu+2)}{\Gamma(\nu-l+\frac{5}{2})}p\int_0^\infty {}_2F_1\left[\begin{matrix}2,\ 2+2\nu\\ \nu-l+5/2\end{matrix};\ -\frac{p}{t}\right]t^{l-\nu-5/2}F\left(\frac{1}{t}\right)dt \tag{4.1}$$

provided that $R(p) > 0$, $R(2+2\nu) > 0$, $R(\mu) > -1$, $R(\mu+2\nu) > -1$, $R(2-\varrho) > 1$,

where $t^{l-\nu-\frac{1}{2}}F\left(\frac{1}{t}\right) = O(t^{\mu})$ for small t *and* $t^{l-\nu-\frac{1}{2}}F\left(\frac{1}{t}\right) = O(t^{\rho})$ for large t.

Proof. If

$$f(p) \doteqdot h(x)$$

$$p^{-2\nu}h(1/p) \doteqdot g(x)$$

and

$$F(p) \doteqdot x^{\frac{1}{2}-\nu-l}g(x)$$

then (Harishanker, 1948)

$$f(p) = p^{\nu-l+\frac{1}{2}}\int_0^\infty e^{-\frac{1}{2}u}u^{l-1}W_{l,\nu}(u)F(p/u)du \tag{4.2}$$

provided that $R(p) > 0$, the Laplace transforms of $|g(t)|$ and $|t^{\frac{1}{2}-l-\nu}g(t)|$ exist, and the integral in (4.2) is absolutely convergent.

In (4.2) if we put $u = pt$ we obtain

$$f(p) = p\int_0^\infty e^{-\frac{1}{2}pt}(pt)^{\nu-\frac{1}{2}}W_{l,\nu}(pt)t^{l-\nu-\frac{1}{2}}F\left(\frac{1}{t}\right)dt \tag{4.3}$$

We have also due to C. B. Rathie (1952, p. 248) the following theorem :—

if
$$f(p) \doteqdot h(v)$$

and
$$p^{2-l}h(p) = p\int_0^\infty e^{-\frac{1}{2}px}(px)^{m-\frac{1}{2}}W_{k,m}(px)g(x)dx$$

then
$$f(p) = \frac{\Gamma(l)\Gamma(l+2m)}{\Gamma(l+m-k+\frac{1}{2})}\,p\int_0^\infty t^{-l}\,{}_2F_1\Big(l, l+2m, l+m-k+\tfrac{1}{2};\ -\frac{p}{t}\Big)g(t)dt \tag{4.4}$$

provided that $R(l) > 0$, $R(l+2m) > 0$, $R(p) \geq 0$, $R(\mu) > -1$, $R(\mu+2m) > -1$, $R(l-\nu) > 1$, when $g(t) = O(t^\mu)$ for small t and $g(t) = O(t^\nu)$ for large t.

Now considering (4.3) with
$$\psi(p) \doteqdot f(x)$$

and with the help of Rathie's theorem, where we take $l = 2$, $k = l$ and $m = \nu$, we immediately obtain the result (4.1).

5. *Example 1.* Let
$$g(x) = \frac{x^{2\nu}}{\Gamma(2\nu+1)} \doteqdot \frac{1}{p^{2\nu}} = p^{-2\nu}h\Big(\frac{1}{p}\Big)$$

$R(p) > 0$, $R(2\nu+1) > 0$,

then
$$x^{\frac{1}{2}-\nu-l}g(x) = \frac{x^{\frac{1}{2}+\nu-l}}{\Gamma(2\nu+1)} \doteqdot \frac{\Gamma(3/2+\nu-l)}{\Gamma(2\nu+1)}\,\frac{1}{p^{\nu-l+\frac{1}{2}}} = F(p)$$

$R(p) > 0$, $R(\nu-l+\frac{3}{2}) > 0$.

And therefore from (4.1) we get
$$\int_0^\infty \frac{1}{t^2}\,{}_2F_1\left[\begin{matrix}2, 2+2\nu\\ \nu-l+\frac{5}{2}\end{matrix};-\frac{p}{t}\right]dt = \frac{(\nu-l+\frac{3}{2})}{(2\nu+1)}\,\frac{1}{p} \tag{5.1}$$

$R(p) > 0$, $R(2\nu+1) \geq 0$ and $R(\nu-l+\frac{3}{2}) > 0$.

Example 2. Let
$$g(x) = e^{-\frac{1}{2}x}K^{2l}_{2n}(\tfrac{1}{2}x) \doteqdot (-1)^{n-l-1}\frac{p^{n-l}}{(1+p)^{n+l+1}} = p^{-2\nu}h\Big(\frac{1}{p}\Big)$$

then
$$h(x) = (-1)^{n-l-1}\frac{x^{1+2l-2\nu}}{(1+x)^{n+l+1}} \doteqdot (-1)^{n-l-1}\frac{e^{\frac{1}{2}p}\Gamma(2l-2\nu+2)}{p^{\frac{1}{2}(l-n-2\nu)}}W_{(\nu-\frac{1}{2}n,\,-3l/2-1),\,\frac{1}{2}(l-n-2\nu+1)}(p) = f(p)\cdot$$

$R(p) > 0$, $R(l-\nu+1) \geq 0$,

and $$f(x) = (-1)^{n-l-1}\frac{e^{\frac{1}{2}x}\Gamma(2l-2\nu+2)}{x^{\frac{1}{2}(l-n-2\nu)}} W_{(\nu-n/2-3l/2-1),\ \frac{1}{2}(l-n-2\nu+1)}$$

$$\doteqdot (-1)^{n-l-1}\frac{\Gamma(2\nu+n-l+1)\Gamma(2l-2\nu+2)}{\Gamma(n+l+8)}\ {}_2F_1\left[\begin{matrix}2\nu+n-l+1,\ 2l-2\nu+2\\ n+l+8\end{matrix};1-p\right]=\psi(p)$$

$R(p) > 0, \quad R(2\nu+n-l+1) > 0.$

Hence due to Shastri's result (1944, theorem III)

$$e^{\frac{1}{2}p}W_{(\nu-n/2-3l/2-1),\ \frac{1}{2}(l-n-2\nu+1)}(p) = \frac{2p^{\frac{1}{2}(l-n+2)}}{\Gamma(2l-2\nu+2)}\int_0^{\infty} t^{-\nu}e^{-\frac{1}{2}t}K_{2\nu}(2\sqrt{pt})K_{2n}^{2l}(\tfrac{1}{2}t)dt \qquad (5.2)$$

where $R(p) > 0,\ R(l-\nu+1) > 0,\ R(\nu) < \frac{1}{2}$ and $l > -1.$

If we take $\frac{1}{2}-\nu-l = 0$ then because of (4 1) we get

$$\frac{\Gamma(2\nu+n-l+1)\,\Gamma(2l-2\nu+2)}{\Gamma(n+l+8)}\,{}_2F_1\left[\begin{matrix}2\nu+n-l+1,\ 2l-2\nu+2;\ 1-p\\ n+l+8\end{matrix}\right]$$

$$= \int_0^{\infty}\frac{1}{t^2}\,{}_1F_0\left[2;\ -\frac{p}{t}\right]\frac{t^{2l+1}}{(1+t)^{n+l+1}}\,dt \qquad (5.3)$$

$R(p) > 0,\ R(l-\nu+1) > 0,\ R(2\nu+n-l+1) > 0,\ l > -1,\ R(l+\nu) > 0,\ n-l+1 > 0.$

And because of (4.2) we get

$$e^{\frac{1}{2}p}\frac{\Gamma(2l-2\nu+2)}{p^{\frac{1}{2}(l-n-2\nu)}}W_{(\nu-\frac{1}{2}n-3l/2-1),\ \frac{1}{2}(l-n-2\nu+1)}(p)$$

$$= p^{n-l}\int_0^{\infty} e^{-\frac{1}{2}u}u^{-(\frac{1}{2}+\nu)}\,W_{\frac{1}{2}-\nu,\ \nu}(u)\frac{u^{2l+1}}{(u+p)^{n+l+1}}du$$

$$R(p) > 0,\ R(l-\nu+1) > 0,\ l > -1. \qquad (5.4)$$

Example 8. Consider

$$g(x) = e^{-x}x^{m}T_m^n(x) \doteqdot \frac{(-1)^n}{p^{2\nu}}\ \frac{p^{n+2\nu+1}}{n\,!(1+p)^{m+n+1}} = \frac{1}{p^{2\nu}}\,h\!\left(\frac{1}{p}\right)$$

$R(p) > 0, \quad R(m+n) > -1,$

where $T_m^n(x)$ is Sonine's polynomial which is a generalisation of the parabolic cylinder function.

Hence,

$$h(x) = \frac{(-1)^n x^{m-2\nu}}{n\,!\,(1+x)^{m+n+1}} \doteqdot (-1)^n\frac{\Gamma(m-2\nu+1)}{n\,!}\,p^{\frac{1}{2}(n+2\nu+1)}e^{\frac{1}{2}p}W_{-\frac{1}{2}(n-2\nu+2m+1),\ -\frac{1}{2}(2\nu+n)}(p) = f(p)$$

$R(p) > 0, \quad R(m-2\nu+1) \geq 0.$

Also

$$f(x) = (-1)^n \frac{\Gamma(m-2\nu+1)}{n\,!} x^{\frac{1}{2}(n+2\nu+1)} e^{\frac{1}{2}x} W_{-\frac{1}{2}(n-2\nu+2m+1),\,-\frac{1}{2}(n+2\nu)}(x)$$

$$\doteqdot (-1)^n \frac{\Gamma(m-2\nu+1)\Gamma(2\nu+n+2)}{n\,!\,\Gamma(n+m+3)}\, {}_2F_1\left[\begin{matrix}2\nu+n+2,\ 2\\ n+m+3\end{matrix};1-p\right] = \psi(p)$$

$R(p) > 0, \quad R(2\nu+n+2) > 0.$

Hence from (4.1), when $\frac{1}{2}-\nu-l = 0$

$$\frac{\Gamma(m-2\nu+1)\Gamma(2\nu+n+2)}{n\,!\,\Gamma(n+m+3)}\, {}_2F_1\left[\begin{matrix}2\nu+n+2,\ 2\\ n+m+3\end{matrix};1-p\right] = \frac{1}{n\,!}\int_0^\infty \frac{1}{t^2}\, {}_1F_0\left[2\,;-\frac{p}{t}\right]\frac{t^m}{(1+t)^{m+n+1}}dt \qquad (5.5)$$

$R(p) > 0, \quad R(\nu+1) > 0, \quad R(m-2\nu+1) \geq 0, \quad R(m+n) > -1.$

And from (4.2) we obtain

$$\frac{\Gamma(m-2\nu+2)}{n\,!}\, p^{\frac{1}{2}(n+2\nu+1} e^{\frac{1}{2}p} W_{-\frac{1}{2}(n-2\nu+m+1),\,-\frac{1}{2}(2\nu+n)}(p)$$

$$= p^{2\nu+n+1}\int_0^\infty e^{-\frac{1}{2}u} u^{-(\nu+\frac{1}{2})} W_{\frac{1}{2}-\nu,\,\nu}\, \frac{u^m}{(p+u)^{m+n+1}}\, du \qquad (5.6)$$

$R(p) > 0, \quad R(m+n) > -1$ and $R(m-2\nu+1) > 0.$

Example 4. This is an illustration of Shastri's result (1944, III theorem) and Harishanker's result (4.2).

We have the operational representation [Chakrabarti, N. K., 1953; p. 67]

$$f(p) = \Gamma(l+n-1)\Gamma(l-n+1)p^{-2l} e^{\frac{1}{2}p} K_{2n}^{2l}(\tfrac{1}{2}p) \doteqdot x^{l-n}(1+x)^{l+n} = h(x).$$

We have also $\dfrac{\Gamma(1+\nu)}{p^\nu} \doteqdot x^\nu \quad R(p) > 0, \quad R(\nu+1) \geq 0.$

So that on term by term interpretation,

$$p^{-2\nu}h\left(\frac{1}{p}\right) = p^{n-l-2\nu}\, {}_1F_0\left[-(n+l)\,;\,-\frac{1}{p}\right] \doteqdot \frac{x^{2\nu+l-n}}{\Gamma(2\nu+l-n+1)}\, {}_1F_1[-(n+l)\,;$$

$$(2\nu+l-n+1)\,;\,-x] = g(x)$$

$R(p) > 0, \quad R(2\nu+l-n+1) > 0.$

Hence because of Shastri's theorem

$$e^{\frac{1}{2}p} K_{2n}^{2l}(\tfrac{1}{2}p) = \frac{2p^{\nu+2l+1}}{\Gamma(l+n+1)\Gamma(l-n+1)}\int_0^\infty \frac{t^{\nu+l-n}}{\Gamma(2\nu+l-n+1)}\, {}_1F_1[-(n+l)\,;(2\nu+l-n-1)\,;\,-t]$$

$$\times K_{2\nu}(2\sqrt{pt})dt \qquad (5.7)$$

where $R(p) > 0, \quad R(2\nu+l-n+1) > 0, \ (l-n+1) > 0, \ l > -1.$

Further
$$\bar{g}(x)x^{\frac{1}{2}-\nu-l} = x^{\nu-n+\frac{1}{2}} \ {}_1F_1[-(n+l) ; (2\nu+l-n+1) ; -x]$$
$$+ \frac{\Gamma(\nu-n+\frac{3}{2})}{p^{\nu-n+\frac{1}{2}}} \ {}_2F_1\left[\begin{matrix} -(n+l), (\nu-n+\frac{3}{2}) \\ (2\nu+l-n+1) \end{matrix} ; -\frac{1}{p}\right]$$

$R(p) > 0, \quad R(\nu-n+\frac{3}{2}) > 0.$

Hence (4.2) gives

$$e^{\frac{1}{2}p}K^{2l}_{2n}(\tfrac{1}{2}p) = \frac{\Gamma(\nu-n+\frac{3}{2})p^{n+l}}{\Gamma(l+n-1)\Gamma(l-n+1)} \int_0^\infty e^{-\frac{1}{2}u}u^{\nu-n+l-\frac{1}{2}}W_{l,\nu}u \ {}_2F_1\left[\begin{matrix} -(n+l), (\nu-n+\frac{3}{2}) \\ (2\nu+l-n+1) \end{matrix} ; -\frac{u}{p}\right]du$$

$R(p) > 0, \quad R(2\nu+l-n+1) \quad R(\nu-n+\frac{3}{2}) > 0, \quad (l-n+1) > 0$ and $l > -1.$

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ON SOME INTEGRALS INVOLVING LEGENDRE FUNCTION, ASSOCIATED LEGENDRE FUNCTION AND JACOBI POLYNOMIALS

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Introduction. Cooke (1924) has evaluated the integral

$$\int_{-1}^{1} (1+x)^p P_n(x)\, dx,$$

where p is a positive integer greater than the integer n. N. G. Shabde later (1945) obtained the value of the above integral for unrestricted values of p and n. In its evaluation he has made use of the following theorem:

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F(x)\, G(x)\, dx$$

where $f(x)$ and $g(x)$ are the Hankel-Transforms of $F(x)$ and $G(x)$ respectively.

This paper gives a simpler proof of the above result and further adds some generalisations of this integral. Besides, some more integrals involving associated Legendre-function and Jacobi Polynomials have also been evaluated. These results are believed to be new.

1. From Mac-Robert (1947, p. 111) we have,

$$P_n(x) = \sum_{r=0}^{\infty} \frac{(-n)_r (n+1)_r}{r!\; r!\; 2^r} (1-x)^r,$$

where n is unrestricted.

Therefore,

$$\int_{-1}^{1} (1+x)^p P_n(x)\, dx = \sum_{r=0}^{\infty} \frac{(-n)_r (n+1)_r}{r!\; r!\; 2^r} \int_{-1}^{1} (1+x)^p (1-x)^r\, dx$$

$$= \sum_{r=0}^{\infty} \frac{(-n)_r (n+1)_r}{r!\; r!\; 2^r}\, 2^{p+r+1} \frac{\Gamma(p+1)\Gamma(r+1)}{\Gamma(p+r+2)}$$

$$= \frac{2^{p+1}}{p+1}\, {}_2F_1\left[\begin{matrix} -n,\; n+1\,; \\ p+2\,; \end{matrix}\; 1\right]$$

$$= \frac{2^{p+1}\{\Gamma(p+1)\}^2}{\Gamma(p+n+2)\,\Gamma(p-n+1)} \tag{1.2}$$

The term-by-term integration in the above process is justified as the series (1.1) is uniformly covergent in the range $(-1 \leq x \leq 1)$ by Weierstrass M-Test.

The same is true in case of the results that follow.

2. Now starting with the result (Mac-Robert 1947, p. 123)

$$P_n^m(x) = \frac{\Gamma(n+m+1)(-1)^{\frac{1}{2}m}}{\Gamma(n-m+1)\, m\,!}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}m} F[-n, n+1 ;\ m+1 ;\ \tfrac{1}{2}(1-x)] \tag{2.1}$$

where m is a positive integer and n is unrestricted, we obtain

$$\int_{-1}^{1} (1-x^2)^{\frac{1}{2}m}(1+x)^p\, P_n^m(x)\, dx$$

$$= (-1)^{\frac{1}{2}m} \frac{2^{p+m+1}}{m\,!} \cdot \frac{\Gamma(m+n+1)\,\Gamma(m+1)\,\Gamma(p+m+1)}{\Gamma(n-m+1)\,\Gamma(p+n+2)\,\Gamma(p+m-n+1)} \tag{2.2}$$

where $p > -m-1$.

Putting $m = 0$ in (2.2) we readily see that

$$\int_{-1}^{1} (1+x)^p\, P_n(x)\, dx = \frac{2^{p+1}\{\Gamma(p+1)\}^2}{\Gamma(p+n+2)\,\Gamma(p-n+1)}.$$

Here it may be remarked that Thosar [1954] has evaluated the integral

$$\int_{-1}^{1} x^l(1-x^2)^{\frac{1}{2}m}\, P_s^m(x)\, dx$$

where l, s and m are positive integers. But this result is not new as it is included in the Barne's (1908) integral

$$\int_{0}^{1} x^\sigma(1-x^2)^{\frac{1}{2}m}\, P_n^m(x)\, dx$$

$$= \frac{(-1)^m\Gamma(\frac{1}{2}+\frac{1}{2}\sigma)\Gamma(1+\frac{1}{2}\sigma)\Gamma(1+m+n)}{2^{m+1}\Gamma(1-m+n)\Gamma(1+\frac{1}{2}\sigma+\frac{1}{2}m-\frac{1}{2}n)\Gamma(\frac{3}{2}+\frac{1}{2}\sigma+\frac{1}{2}m+\frac{1}{2}n)}$$

$Re\ (\sigma) > -1$ and m is a positive integer.

3. Again, if we take the expansion of $P_n^m(x)$ for unrestricted values of m and n (Mac-Robert, 1947, p. 314)

$$P_n^m(x) = \frac{(-1)^{\frac{1}{2}m}}{\Gamma(1-m)}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}m} F(-n, n+1 ;\ 1-m ;\ \tfrac{1}{2}(1-x)) \tag{3.1}$$

we obtain

$$\int_{-1}^{1} (1+x)^p(1-x^2)^{-\frac{1}{2}m} P_n^m(x)\, dx = (-1)^{\frac{1}{2}m} 2^{p-m+1} \frac{\Gamma(p+1)\Gamma(p-m+1)}{\Gamma(p+n-m+2)\Gamma(p-m-n+1)} \tag{3.2}$$

where $p > m-1$

From (3.2), by putting $m = 0$, we obtain the same result (1.2)

4. Nezt, from Szegö (1948, p. 6) we have

$$P_n^{\alpha,\beta}(x) = \binom{n+\alpha}{n} F\left[\begin{matrix} -n,\ n+\alpha+\beta+1\,; \\ \alpha+1\,; \end{matrix}\ \tfrac{1}{2}(1-x)\right] \tag{4.1}$$

Hence we obtain

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{p} P_n^{\alpha,\beta}(x)\,dx$$

$$= \binom{n+\alpha}{n} 2^{\alpha+p+1} \frac{\Gamma(p+1)\Gamma(\alpha+1)\Gamma(p-\beta+1)}{\Gamma(\alpha+p+n+2)\Gamma(p-\beta+n+1)} \tag{4.2}$$

where $\alpha > -1$, $\beta > -1$ and $p > \beta - 1$

in which putting $\alpha = \beta = 0$ we again get (1.2).

5. The evaluation of the Integral

$$\int_{-1}^{1} (1+x)^{p} P_n(x)\, P_m(x)\,dx$$

in which p and n are unrestricted while m is a positive integer may be done on similar lines by expressing $P_m(x)$ in a series of powers of $(1+x)$, thus

$$P_m(x) = (-1)^m \sum_{r=0}^{m} \frac{(-m)_r (m+1)_r}{r!\ r!\ 2^r} (1+x)^r,$$

therefore

$$\int_{-1}^{1} (1+x)^{p} P_n(x)\, P_m(x)\,dx = (-1)^m \sum_{r=0}^{m} \frac{(-m)_r (1+m)_r}{r!\ r!\ 2^r} \int_{-1}^{1} (1+x)^{p+r} P_n(x)\,dx$$

$$= \frac{(-1)^m 2^{p+1}\{\Gamma(p+1)\}^2}{\Gamma(p+n+2)\,\Gamma(p-n+1)}\ {}_4F_3\left[\begin{matrix} -m,\ m+1,\ p+1,\ p+1\,; \\ p+n+2,\ p-n+1,\ 1\,; \end{matrix}\ 1\right]$$

6. Let us now evaluate the integral

$$\int_{-1}^{1} (1-x)^{p} P_n(x)\,dx$$

Starting with

$$P_n(x) = \sum_{r=0}^{m} \frac{(-n)_r (n+1)_r}{r!\ r!\ 2^r} (1-x)^r$$

We obtain

$$\int_{-1}^{1} (1-x)^{p} P_n(x)\,dx = \sum_{r=0}^{\infty} \frac{(-n)_r (n+1)_r}{r!\ r!\ 2^r}\, 2^{p+r+1} \frac{\Gamma(p+r+1)}{\Gamma(p+r+2)}$$

$$= \frac{2^{p+1}}{p+1}\ {}_3F_2\left[\begin{matrix} -n,\ n+1,\ p+1\,; \\ 1,\ p+2\,; \end{matrix}\ 1\right] \tag{6.1}$$

Here p and n are unrestricted.

When n is a positive integer, ${}_3F_2$ can be evaluated by Saalschütz's theorem (Erdélyi, 1953, p. 188) and since $P_n(-x) = (-1)^n P_n(x)$, we obtain

$$\int_{-1}^{1}(1+x)^p P_n(x)\,dx = \frac{2^{p+1}}{p+1}\frac{(p-n+1)_n(1)_n}{(p+2)_n(-n)_n}(-1)^n$$

$$= \frac{2^{p+1}\{\Gamma(p+1)\}^2}{\Gamma(p+n+2)\,\Gamma(p-n+1)}$$

as before.

Next, proceeding in a similar way we get

$$\int_{-1}^{1}(1-x^2)^p P_n(x)\,dx = \frac{2^{p+1}\{\Gamma(p+1)\}^2}{\Gamma(p+2)}\,{}_3F_2\left[\begin{matrix}-n,\ n+1,\ p+1\,; \\ 1,\ 2p+2\,;\end{matrix}\ 1\right]$$

which in virtue of Watson's theorem (Erdélyi, 1953, p. 189), becomes

$$= \frac{2^{p+1}\{\Gamma(p+1)\}^3\Gamma(\frac{1}{2})\Gamma(p+\frac{3}{2})}{\Gamma(2p+2)\Gamma(\frac{1}{2}-\frac{1}{2}n)\Gamma(1+\frac{1}{2}n)\Gamma(p+\frac{3}{2}+\frac{1}{2}n)\Gamma(p+1-\frac{1}{2}n)} \tag{6.2}$$

where $Re(p) > -1$

7. We shall now make use of the integral in (6.2) to obtain summation of finite series involving associated Legendre function of the second kind.

Thus when n is a positive integer, we have

$$(1-x^2)^m = \sum_{r=0}^{m} a_{2r} P_{2r}(p) \tag{7.1}$$

where

$$a_{2r} = \frac{4r+1}{2}\int_{-1}^{1}(1-x^2)^m P_{2r}(x)\,dx$$

$$= \frac{(4r+1)2^{2m}\{\Gamma(m+1)\}^3\Gamma(\frac{1}{2})\Gamma(m+\frac{3}{2})}{\Gamma(2m+2)\Gamma(\frac{1}{2}-r)\Gamma(1+r)\Gamma(m+r+\frac{3}{2})\Gamma(m-r+1)}$$

Dividing (7.1) by $(y-x)^{2m+2}$ and integrating with respect to x between the limits $(-1, 1)$ we obtain (Thosar, 1954, p p. 53-54)

$$-2^{2m-2}(m\,!)^2(y^2-1)^{-\frac{1}{2}} = \sum_{r=0}^{m} a_{2r} Q_{2r}^{2m+1}(y), \quad (y > 1) \tag{7 2}$$

Again, dividing (7.1) by $(y-x)^s$ and integrating with respect to x between the limits $(-1, 1)$ we get (Thosar, 1954, p.p. 53-57)

$$\frac{2^{2m+1}(m\,!)^2}{(2m+1)\,!}(y+1)^{-s}F(s,\ m+1\,;\ 2m+2\,;\ 2/(y+1))$$

$$= \sum_{r=0}^{m}(-1)^{s-1}2/(s-1)\,!\ (y^2-1)^{\frac{1}{2}(1-s)}a_{2r}\,Q_{2r}^{s-1}(y), \quad (y > 1,\ s > 1) \tag{7.3}$$

In (7.3) if $s = n+m+1$, then (Mac-Robert, 1947, p. 365 and Thosar, 1954, p. 53)

$$Q_n^m(y) = \frac{(-1)^{n+m}(y^2-1)^{-\frac{1}{2}m}}{m!\,2^m} \sum_{r=0}^{m} a_{2r}\, Q_{2r}^{n+m}(y) \tag{7.4}$$

where m and n are positive integers and $y > 1$.

8. We now take

$$\int_{-1}^{1} x^{p-q} P_{p+m}^m(x)\, P_{q+m}^m(x)\, dx$$

where p and q are positive integers and $p > q$.

From Thosar (1954, p. 55)

$$x^{n-m}(1-x^2)^{\frac{1}{2}m} = (n-m)! \sum_{r=0}^{\frac{1}{2}(n-m)} \frac{(2n-4r+1)2^{n-2r}(n-r)!}{r!\,(2n-2r+1)!} P_{n-2r}^m(x) \tag{8.1}$$

where m is an integer $\leq n$.

Also from Bailey (1939)

$$(1-x^2)^{-\frac{1}{2}m} P_{p+m}^m(x)\, P_{q+m}^m(x) = \sum_{r=0}^{(\min p,\, q)} \frac{2^{m-1}(p+q+2m-r)!\,\Gamma(m+r+\frac{1}{2})}{\pi^{\frac{1}{2}} r!\,\Gamma(m+\frac{1}{2})\,(p-r)!\,(q-r)!}$$

$$\times \frac{\Gamma(p+m-r+\frac{1}{2})\Gamma(q+m-r+\frac{1}{2})(p+q-2r)!\,(2p+2q+2m-4r+1)}{\Gamma(p+q+m-r+\frac{3}{2})\,(p+q+2m-2r)!} P_{p+q+m-2r}^m(x) \tag{8.2}$$

Multiplying (8.1) and (8.2) and putting $n = p-q+m$, $(p > q)$ we obtain, on using the orthogonal property

$$\int_{-1}^{1} x^{p-q} P_{p+m}^m(x)\, P_{q+m}^m(x)\, dx$$

$$= \frac{2^{p-q+2m}}{\pi^{\frac{1}{2}} q!} \cdot \frac{(p-q+m)!\,\Gamma(m+q+\frac{1}{2})\,\Gamma(p+m-q+\frac{1}{2})(p+2m)!}{\Gamma(p+m+\frac{3}{2})} \tag{8.3}$$

putting $m = 0$, we find that

$$\int_{-1}^{1} x^{p-q} P_p(x)\, P_q(x)\, dx = \frac{2^{p-q}}{\pi^{\frac{1}{2}}} \cdot \frac{(p-q)!\,p!}{q!} \cdot \frac{\Gamma(q+\frac{1}{2})\,\Gamma(p-q+\frac{1}{2})}{\Gamma(p+\frac{3}{2})} \tag{8.4}$$

Further more, if $p = q$ we obtain the well-known result

$$\int_{-1}^{1} [P_p(x)]^2\, dx = \frac{2}{2p+1}$$

9. We now consider the integral

$$\int_{-1}^{1} (1+x)^{p-q} P_{p+m}^m(x)\, P_{q+m}^m(x)\, dx$$

where p, q and m are positive integers and $p > q$

Taking

$$(1+x)^s = \sum_{r=0}^{s} a_r \, P_r(x) \tag{9.1}$$

where

$$a_r = \frac{2r+1}{2} \int_{-1}^{1} (1+x)^s \, P_r(x) \, dx$$

$$= \frac{2^s \cdot s\,!\,(2r+1)}{(s+r+1)\,!\,(s-r)\,!}$$

Differentiating (9.1) m-times and multiplying by $(1-x^2)^{\frac{1}{2}m}$, we get with Ferrar's definition,

$$\frac{s\,!}{(s-m)\,!} (1-x^2)^{\frac{1}{2}m} (1+x)^{s-m} = \sum_{r=m}^{s} a_r P_r^m(x) \tag{9.2}$$

Let $s = p-q+m$, so that from (8.2) and (9.2) we obtain

$$\int_{-1}^{1} (1+x)^{p-q} p_{p+m}^{m}(x) p_{q+m}^{m}(x) px$$

$$= \frac{2^{p-q+2m}(p-q+m)\,!\,(p+2m)\,!\,\Gamma(m+q+\frac{1}{2})\Gamma(m+p-q+\frac{1}{2})}{\pi^{\frac{1}{2}} q\,!\,(2p-2q+2m)\,!\,\Gamma(p+m+\frac{3}{2})} \tag{9.8}$$

Putting $m = 0$, we get

$$\int_{-1}^{1} (1+x)^{p-q} P_p(x) P_q(x) dx = \frac{2^{p-q}(p-q)\,!\,p\,!\,\Gamma(q+\frac{1}{2})\Gamma(p-q+\frac{1}{2})}{\pi^{\frac{1}{2}} q\,!\,(2p-2q)\,!\,\Gamma(p+\frac{3}{2})} \tag{9.4}$$

Further when $p = q$

$$\int_{-1}^{1} [P_p(x)]^2 dx = \frac{2}{2p+1}\,.$$

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NORMALISATION OF FREQUENCY FUNCTIONS*

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Introduction. It has been found in general that most of the important sampling distributions depend on the sample size n in such a manner that as n tends to infinity distribution tends to normality. Thus for large n we may assume the sampling distribution to be normal to a sufficient degree of approximation, but for small and even for moderate values of n this assumption of normality does not hold good. It must be noted that the tendency towards normality does not depend exclusively on the values of n, cases may arise in which this depends on other factors, *viz.* in the distribution of classical D^2-Statistic, it tends towards normality for increasing values of p (no. of variates not related to n) and $\delta(=n\Delta^2$ related to $n)$; (Bose, 1932).

In the first part of the paper an attempt has been made to assess the amount of error committed in replacing any sampling distribution by the so-called Normal Distribution for moderate values of n and other factors and to construct a (β_1, β_2) diagram corresponding to some pre assigned error. If the amount of error committed is within permissible limits then the replacement is valid for all practical purposes. Some applications have been given in the second part of this paper. A preliminary study was made by Bose and Rao (1948).

In this connection I would like to mention why these particular approach is important. While constructing the tables for classical D^2-statistic it became necessary to know the (p, δ). combinations for which we can use the Normal tables (Bose 1947), provided we ignore small errors. Present method helps us to assess this to a satisfactory extent. The method is general and has been applied to many well known sampling distributions. Fisher and Cornish (1937) made a similar study.

PART I—METHOD

1.1. Error in replacing any distribution function by Normal distribution function.

Let ξ be the variable whose sampling distribution we are concerned with and let m, σ and μ_r be its mean, standard deviation and moment of order r respectively. Then

$$x = (\xi - m)/\sigma$$

will be corresponding standardized variable, and the moment of order r of x will be μ_r/σ^r. Let $F(x)$ and $f(x)$ be the corresponding distribution and frequency functions of x respectively. Also let the distribution function and frequency functions for the Normal distribution be as follows

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* The paper was discussed at the International Mathematical Congress at Amsterdam 1954 and the abstract was published in the Proceedings of the Congress.

$$\Theta(x) = \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{x} e^{-\frac{1}{2}x^2}dx\,;\ \varphi(x) = \Theta'(x) = \frac{1}{(2\pi)^{\frac{1}{2}}}e^{-\frac{1}{2}x^2}$$

We may write

$$F(x) = \Theta(x)+R(x) \tag{1.1}$$

and

$$f(x) = \varphi(x)+r(x) \tag{1.2}$$

Then the above discussions imply that $R(x)$ and $r(x)$ are small for large values of n, so that $\Theta(x)$ and $\varphi(x)$ may be regarded as first approximations to $F(x)$ and $f(x)$ respectively.

The next question then automatically arises whether it is possible to find some convenient expansion for the remainder term in order to get more accurate approximations. Two different approaches are well-known and may be mentioned in this connection. One is Gram-Charlier series of Type A and the other is Edgeworth's series. By both methods $f(x)$ may be written as

$$f(x) = \varphi(x)-\frac{1}{3\,!}\mu_3\varphi^3(x)+\frac{1}{4\,!}(\mu_4-3)\varphi^4(x)+\frac{10}{6\,!}\mu_3{}^2\varphi^6(x)$$
$$-\frac{1}{5\,!}(\mu_5-10\mu_3)\varphi^5(x)-\frac{35}{7\,!}\mu_3(\mu_4-3)\varphi^7(x)-\frac{280}{9\,!}\mu_3{}^3\varphi^9(x)+\ldots \tag{1.3}$$

where x is expressed in standard measure and $\varphi(x) = \frac{1}{(2\pi)^{\frac{1}{2}}}e^{-\frac{1}{2}x^2}$. The asymptotic properties of the series has been investigated by Cramér (1928, 1937)

Retaining terms up to order $O(n^{-1})$ we can write down

$$f(x) = \varphi(x)-\frac{\beta_1{}^{\frac{1}{2}}}{6}\varphi^3(x)+\frac{\beta_2-3}{24}\varphi^4(x)+\frac{\beta_1}{72}\varphi^6(x) \tag{1.4}$$

where x is expressed in standard measure.

$$= \left[1+\frac{\beta_1{}^{\frac{1}{2}}}{6}H_3(x)+\frac{\beta_2-3}{24}H_4(x)+\frac{\beta_1}{72}H_6(x)\right]\varphi(x) \tag{1.5}$$

where H_j is the Hermite polynomial defined as $(-1)^j\varphi^j(x) = H_j(x)\varphi(x)$

The error we commit in the probability integral if we replace $f(x)$ by $\varphi(x)$ only is

$$\int_{-\infty}^{x}f(x)dx-\int_{-\infty}^{x}\varphi(x)dx = -\left[\frac{\beta_1{}^{\frac{1}{2}}}{6}H_2(x)+\frac{\beta_2-3}{24}H_3(x)+\frac{\beta_1}{72}H_5(x)\right]\varphi(x) \tag{1.6}$$

Here, of course, we have used the series for $f(x)$ up to terms of order $O(n^{-1})$ only.

The magnitude of error in the probabity integral will be different for different values of x. We try find out that value of x for which the correspondiug absolute error in the probability integral is maximum. For this we require the following conditions.

$$\left[\frac{\beta_1^{\frac{1}{2}}}{6}H_3(x)+\frac{\beta_2-3}{24}H_4(x)+\frac{\beta_1}{7}H_6(x)\right]\varphi(x)=0 \tag{1.7}$$

$$\left[\frac{\beta_1^{\frac{1}{2}}}{6}H_4(x)+\frac{\beta_2-3}{24}H_5(x)+\frac{\beta_1}{72}H_7(x)\right] \tag{1.8}$$

is positive or negative according as the above error is positive or negative

The maximum error will then be obtained easily from (1.6)

The absolute error at any value of x will be less than this maximum absolute error or in other words the absolute error in the probability integral will never be greater than this (maximum absolute error) for any value of x if we replace any distribution by the Normal distribution.

For any distribution (not normal) it is thus possible by virtue of equations (1.6), (1.7) and (1.8) to construct a table giving the maximum absolute error in the probability integral for different combinations of β_1, β_2 or for different values of n.

1.2. Construction of (β_1, β_2) diagram corresponding to some pre-assigned error. It has been observed in Section I that the error we commit in the probability integral if we replace $f(x)$ by $\varphi(x)$ only, is

$$-\left[\frac{\beta_1^{\frac{1}{2}}}{6}H_2(x)+\frac{\beta_2-3}{24}H_3(x)+\frac{\beta_1}{72}H_5(x)\right]\varphi(x)$$

And the criteria to be satisfied in order that this error in absolute value is maximum are

(*i*) $$\frac{\beta_1^{\frac{1}{2}}}{6}H_3(x)+\frac{\beta_2-3}{24}H_4(x)+\frac{\beta_1}{72}H_6(x)=0$$

(*ii*) $$\frac{\beta_1^{\frac{1}{2}}}{6}H_4(x)+\frac{\beta_2-3}{24}H_5(x)+\frac{\beta_1}{72}H_7(x)$$

is positive or negative according as the above error is positve or negative.

Let the pre-assigned error be $+\alpha$. It is then required to find out the combination of (β_1, β_2) values such that the following conditions are satisfied.

$$-\left[\frac{\beta_1^{\frac{1}{2}}}{6}H_2(x)+\frac{\beta_2-3}{24}H_3(x)+\frac{\beta_1}{72}H_5(x)\right]\varphi(x)=+\alpha \tag{2.1}$$

$$\frac{\beta_1^{\frac{1}{2}}}{6}H_3(x)+\frac{\beta_2-3}{24}H_4(x)+\frac{\beta_1}{72}H_6(x)=0 \tag{2.2}$$

$$\frac{\beta_1^{\frac{1}{2}}}{6}H_4(x)+\frac{\beta_2-3}{24}H_5(x)+\frac{\beta_1}{72}H_7(x) \quad \text{is positive} \tag{2.3}$$

From (2.2) we have

$$\frac{\beta_2-3}{24}=-\left[\frac{\beta_1^{\frac{1}{2}}}{6}\,\frac{H_3(x)}{H_4(x)}+\frac{\beta_1}{72}\,\frac{H_6(x)}{H_4(x)}\right] \tag{2.4}$$

Or

$$\beta_2=3-4\beta_1^{\frac{1}{2}}\,\frac{H_3(x)}{H_4(x)}-\frac{\beta_1}{3}\,\frac{H_6(x)}{H_4(x)} \tag{2.5}$$

Then from (2.1) and (2.4)

$$\left[\frac{\beta_1^{\frac{1}{2}}}{6}\left\{\frac{H_3^2(x)}{H_4(x)}-H_2(x)\right\}+\frac{\beta_1}{72}\left\{\frac{H_6(x)H_3(x)}{H_4(x)}-H_5(x)\right\}\right]\varphi(x)=\alpha \tag{2.6}$$

We can solve (2.6) for $(\beta_1)^{\frac{1}{2}}$ treating this equation as a quadratic equation in $(\beta_1)^{\frac{1}{2}}$ (Imaginary values are to be neglected as β_1 cannot be negative). Then $\beta_1=[(\beta_1)^{\frac{1}{2}}]^2$ and β_2 will be obtained from (2.5). After this we are to see whether criterion (2.3) is satisfied or not.

For pre-assigned negative error say $-\alpha$, the equations (2.1), (2.3) and (2.6) will be replaced by (2.7), (2.8) and (2.9) respectively

$$-\left[\frac{\beta_1^{\frac{1}{2}}}{6}\,H_2(x)+\frac{\beta_2-3}{24}\,H_3(x)+\frac{\beta_1}{72}H_5(x)\right]\varphi(x)=-\alpha \tag{2.7}$$

$$\frac{\beta_1^{\frac{1}{2}}}{6}\,H_4(x)-\frac{\beta_2-3}{24}\,H_5(x)+\frac{\beta_1}{72}H_7(x)\quad\text{is negative} \tag{2.8}$$

$$\left[\frac{\beta_1^{\frac{1}{2}}}{6}\left\{\frac{H_3^2(x)}{H_4(x)}-H_2(x)\right\}+\frac{\beta_1}{72}\left\{\frac{H_6(x)H_3(x)}{H_4(x)}-H_5(x)\right\}\right]\varphi(x)=-\alpha \tag{2.9}$$

However the combination of β_1, β_2 values will remain the same for $+\alpha$ and $-\alpha$, only the sign of $(\beta_1)^{\frac{1}{2}}$ will be different for the two cases.

For a fixed value of α and for different values of x we can get a series of values for β_1 and (β_2-3). We may then plot the values of β_2-3 and β_1 on a graph paper taking (β_2-3) along the x-axis and β_1 along the y-axis and then draw the curve. Whenever any values of β_1 and β_2-3 will fall within the curve, it will mean that the difference of the probability integral of the given curve and normal curve will not exceed the given amount α in absolute value.

In this paper in the series representation terms upto the $O(n^{-1})$ has been retained If more accurate results are necessary the same method can be utilised by retaining higher order terms. A general discussion on this has been given by Kendall (1943).

PART II. APPLICATOIN.

2.1. We have calculated the maximum error in the incomplete probability integrals of t, χ^2, $(2\chi)^{\frac{1}{2}}$ D^2 (classical) distributions for various values of n. In each case we have retained terms up to $O(n^{-1})$. It is to be noted thet the results are not very accurate for small values of n.

(i) *t-distribution*

t-distribution is given by

$$dF = \frac{\Gamma(\frac{1}{2}n+\frac{1}{2})}{\Gamma(n/2)\,\Gamma(\frac{1}{2})n^{\frac{1}{2}}}\,\frac{dt}{(1+t^2/n)^{\frac{1}{2}(n+1)}}$$

with

$$\text{d.f.} = n$$

$$\text{mean} = 0$$

$$\text{s.d} = \left(\frac{n}{n-2}\right)^{\frac{1}{2}}$$

$$\beta_1 = 0,\ \beta_2 = \frac{3(n-2)}{(n-4)}$$

Hence if we replace t-distribution by Normal Distribution the error in the probability integral will be

$$-\left[\frac{H_3(x)}{4(n-4)}\right]\varphi(x)$$

And the criteria to be satisfied in order this error in absolute value is maximum are

(i) $$\frac{H_4(x)}{4(n-4)}\,\varphi(x) = 0$$

(ii) $$\frac{H_5(x)}{4(n-4)}$$

is positive or negative according as the above error is positive or negative.

Consulting Hermite polynomial values the following table has been prepared.

Table 1. Maximum possisle error for t-distribution

n	Maximum error	n	Maximum error
5	·137646	60	·002458
10	·022041	70	·002086
15	·012513	80	·001811
20	·008603	90	·001601
25	·006555	100	·001434
30	·005294	120	·001187
35	·004440	140	·001012
40	·003824	160	·000882
45	·003357	180	·000769
50	·002992	200	·000692

From the table it is easily observed that for values of n equal to or greater then 20, t-distribution may be replaced by the normal distribution allowing 1% error.

(*ii*) *χ^2-distribution.*

χ^2-distribution is given by

$$dF = \frac{1}{\Gamma(n/2)2^{\frac{1}{2}n}}\, e^{-\frac{1}{2}\chi^2}(\chi^2)^{\frac{1}{2}(n-2)}d\chi^2$$

$$\text{d.f.} = n$$

$$\text{mean} = n$$

$$\text{s.d.} = (2n)^{\frac{1}{2}}$$

$$\beta_1 = 8/n$$

$$\beta_2 = 3+12/n$$

Hence if we replace χ^2-distribution by Normal Distribution the error in the probability integral will be

$$-\left[\frac{2^{\frac{1}{2}}}{3n^{\frac{1}{2}}}H_2(x)+\frac{1}{2n}H_3(x)+\frac{1}{9n}H_5(x)\right]\varphi(x)$$

And the criteria to be satisfied in order that this error in absolute value is maximum are

(*i*) $$\frac{2^{\frac{1}{2}}}{3n^{\frac{1}{2}}}H_3(x)+\frac{1}{2n}H_4(x)+\frac{1}{9n}H_6(x) = 0$$

(*ii*) $$\frac{2^{\frac{1}{2}}}{3n^{\frac{1}{2}}}H_4(x)+\frac{1}{2n}H_5(x)+\frac{1}{9n}H_7(x)$$

is positive or negative according as the above error is positive of negative.

Consulting Hermite polynomial values the following table has been prepared.

Table 2. Maximum possible error χ^2-distribution.

n	Maximum error	n	Maximum error
5	·084447	60	·024286
10	·059592	80	·021031
15	·048625	100	·018810
20	·042095	150	·015357
25	·037643	200	·013299
30	·034359	300	·010859
40	·029751	500	·008411
50	·026607	1000	·005947

From the table it is easily observed that for values of n equal to or greater than 200 χ^2-distribution may be replaced by the normal distribution allowing 1% error.

(*iii*) *$(2\chi^2)^{\frac{1}{2}}$ distribution.*

The $(2\chi^2)^{\frac{1}{2}}$-distribution is given by

$$dF = \frac{1}{\Gamma(n/2)2^{n-1}}\, e^{-\frac{1}{2}((2\chi^2)^{\frac{1}{2}})^2}((2\chi^2)^{\frac{1}{2}})^{n-1}d(2\chi^2)^{\frac{1}{2}}$$

$$\begin{aligned}&\text{d.f.} = n\\ &\left.\begin{aligned}&\text{mean} = (2n-1)^{\frac{1}{2}}\\ &\text{s.d} = (1-\tfrac{1}{4n})^{\frac{1}{2}}\\ &\beta_1 = 1/2n,\ \beta_2 = 3.\end{aligned}\right\}\ \text{using approximations upto } O(n^{-1})\end{aligned}$$

Hence if we replace $(2X^2)^{\frac{1}{2}}$ -distribution by Normal distribution the error in the probability integral will be

$$-\left[\frac{1}{6(2n)^{\frac{1}{2}}} H_2(x) + \frac{1}{144n} H_5(x)\right] \varphi(x)$$

And the criteria to be satisfied in order that this error in absolute value is maximum are

(i) $$\frac{1}{6(2n)^{\frac{1}{2}}} H_3(x) + \frac{1}{144n} H_6(x) = 0$$

(ii) $$\frac{1}{6(2n)^{\frac{1}{2}}} H_4(x) + \frac{1}{144n} H_7(x)$$

is positive or negative according as the above error is positive or negative.

Consulting Hermite polynomial values the following table has been prepared.

Table 3. Maximum possible error for $(2X^2)^{\frac{1}{2}}$ *distribution.*

n	Maximum error	n	Maximum error
5	·021556	60	·006083
10	·015058	80	·005265
15	·012243	100	·004708
20	·010580	150	·003842
25	·009452	200	·003327
30	·008621	300	·002716
40	·007458	500	·002103
50	·006666	1000	·001487

From the table it is easily observed that for values of n equal to or greater then 25 $(2X^2)^{\frac{1}{2}}$-distribution may be replaced by the normal distribution allowing 1% error. It is also verified that $(2X^2)^{\frac{1}{2}}$-approaches normal distribution more rapidly than X^2-distribution. This table also verifies Fisher's remark that for values of n larger than 30 the expression $(2X^2)^{\frac{1}{2}} - (2n-1)^{\frac{1}{2}}$ may be used as a normal deviate with unit variance. Fisher (1948).

(iv) *Classical D^2-distribution.*

The D^2-distribution is given by

$$dF = \frac{\bar{n}p}{4}\left(\frac{\bar{n}D^2+2}{\bar{n}\Delta^2}\right)^{(p-2)/4} e^{-\frac{1}{4}\bar{n}p(\Delta^2+D^2+2/\bar{n})} I_{(p-2)/2}\left[\frac{\bar{n}p}{2}\sqrt{\left\{\Delta^2\left(D^2+\frac{2}{\bar{n}}\right)\right\}}\right] d(D^2)$$

$$= \frac{\bar{n}p}{4}\left(\frac{\bar{n}D^2+2}{\delta}\right)^{(p-2)/4} e^{-\frac{1}{4}p(\delta+\bar{n}D^2+2)} I_{(p-2)/2}\left[\frac{p}{2}\sqrt{\left\{\bar{n}\delta\left(D^2+\frac{2}{\bar{n}}\right)\right\}}\right] d(D^2)$$

where $\delta = \bar{n}\Delta^2$

$$\text{with Mean} = \Delta^2$$

$$\text{s.d.} = \left(\frac{8}{pn^2}\right)^{\frac{1}{2}}(\delta+1)^{\frac{1}{2}}$$

$$\beta_1 = \frac{2}{p}\,\frac{(3\delta+2)^2}{(\delta+1)^3}\;;\;\beta_2 = 3+\frac{12}{p}\,\frac{(2\delta+1)}{(\delta+1)^2}$$

Hence if we replace D^2-distribution by Normal distribution the error in the probability integral will be

$$-\left[\frac{1}{3(2p)^{\frac{1}{2}}}\,\frac{3\delta+2}{(\delta+1)^{3/2}}\,H_2(x)+\frac{(2\delta+1)}{2p\,(\delta+1)^2}H_3(x)+\frac{1}{36p}\,\frac{(3\delta+2)^2}{(\delta+1)^3}\,H_5(x) = 0\right.$$

And the criteria to be satisfied in order that this error in absolute value is maximum are

$$(i)\qquad \frac{1}{3(2p)^{\frac{1}{2}}}\,\frac{3\delta+2}{(\delta+1)^{3/2}}\,H_3(x)+\frac{1}{2p}\,\frac{2\delta+1}{(\delta+1)^2}H_4(x)+\frac{1}{36p}\,\frac{(3\delta+2)^2}{(\delta+1)^3}\,H_6(x) = 0$$

$$(ii)\qquad \frac{1}{3(2p)^{\frac{1}{2}}}\,\frac{3\delta+2}{(\delta+1)^{3/2}}\,H_4(x)+\frac{1}{2p}\,\frac{2\delta+1}{(\delta+1)^2}H_5(x)+\frac{1}{36p}\,\frac{(3\delta+2)^2}{(\delta+1)^3}\,H_7(x)$$

is positive or negative according as the above error is positive or negative.

Consulting Hermite polynomial values the following table has been prepared.

Table 4. Maximum possible error for D^2-distribution.

$\delta =$	200	300	400	600	800	1000	2000	3000
$p =$								
1	·019882	·016252	·014081	·011504	·009965	·008915	·006805	·005149
2	·014053	·011488	·009955	·008133	·007046	·006303	·004458	·003641
3	·011472	·009379	·008128	·006640	·005752	·005146	·003640	·002973
4	·009934	·008122	·007039	·005751	·004932	·004457	·003152	·002574
5	·008885	·007264	·006295	·005143	·004463	·003986	·002820	·002303
6	·008111	·006631	·005747	·004695	·004067	·003689	·002574	·002102

If we put
$$\lambda = \left(\frac{np\Delta^2}{2}\right)^{\frac{1}{2}} = \left(\frac{p\delta}{2}\right)^{\frac{1}{2}}$$

(in accordance with the notation in tables of D^2-distribution Bose (1947, 1950)) it is easily observed from the above table that for values of λ equal to or greater than 20 D^2-distribution may be replaced by the normal distribution allowing 1% error.

2.2. For a numerical example in connection with (β_1, β_2) diagram we have considered $\alpha = \pm{\cdot}01$ The values of x, $\beta_1^{\frac{1}{2}}$, β_1 and (β_2-3) are given below. We have considered the values of x from $-{\cdot}7$ to $+{\cdot}7$ so that the error is maximum at some point in the middle half of the normal curve. We might consider the values of x beyond this region and could get corresponding values but that would mean that the maximum error would correspond to a point in the tailends—that case is of no interest to us for in that case the replacement of the given curve by a normal one will be meaningless.

Table 5. Values of β_1 *and* (β_2-3) *for* $\alpha=\pm{}^{\cdot}01$.

$\alpha=+{}^{\cdot}01$		$\alpha=-{}^{\cdot}01$		$\alpha=\pm{}^{\cdot}01$	
x	$\beta_1^{\frac12}$	x	$\beta_1^{\frac12}$	β_1	β_2-3
0	·150398	0	—·150398	·022618	·037625
·1	·148116	—·1	—·148116	·021938	·096446
·2	·141296	—·2	—·141296	·019965	·153025
·3	129443	—·3	—·129443	·016755	·208195
·4	·111850	—·4	—·111850	·012510	·262566
·5	·087659	—·5	—·087659	·007684	·316217
·6	·056257	—·6	— 056257	·003165	·368379
·7	·017876	—·7	—·017876	·000320	·417368
—·1	·148116	1	—·148116	·021938	—·024058
—·2	·141042	·2	—·141042	·019893	—·089187
—·3	·128722	·3	—·128722	016569	—·157349
—·4	110623	·4	— 110623	·012237	—·227200
—·5	·086264	·5	—·086264	·007441	—·296233
—·6	·055328	·6	—·055328	·003061	—·360815
—·7	·017739	·7	—·017739	·000315	—·416675

The releavant graph for the above case is shown below.

2.3. Auxiliary Table of the values of the Hermite-polynomials. Hermite polynomials are defined as

$$H_j(x)\varphi(x) = (-1)^j\varphi^j(x) = \left(-\frac{d}{dx}\right)^j\varphi(x) \quad \text{where } \varphi(x) = \frac{1}{(2\pi)^{\frac12}}\bar{e}^{\frac12 x^2}$$

The following general expression for $H_j(x)$ can be easily obtained

$$H_j(x) = x^j - \frac{j^{[2]}}{2.1!}x^{j-2} + \frac{j^{[4]}}{2^2\cdot 2!}x^{j-4} - \frac{j^{[6]}}{2^3\cdot 3!}x^{j-6} + \dots$$

Whence

$$H_0 = 1$$

$$H_1 = x$$

$$H_2 = x^2-1$$

$$H_3 = x^3-3x$$

$$H_4 = x^4-6x^2+3$$

$$H_5 = x^5-10x^3+15x$$

$$H_6 = x^6-15x^4+45x^2-15$$

$$H_7 = x^7-21x^5+105x^3-105x$$

The very important recurrence relation that is also satisfied

$$H_r(x) = xH_{r-1}(x)-(r-1)H_{r-2}(x)$$

For our study we had to calculate the values of the 1st 8 Hermite polynomials for some different values of x. The values are given in the tables 6 and 7 which are correct to six decimal places.

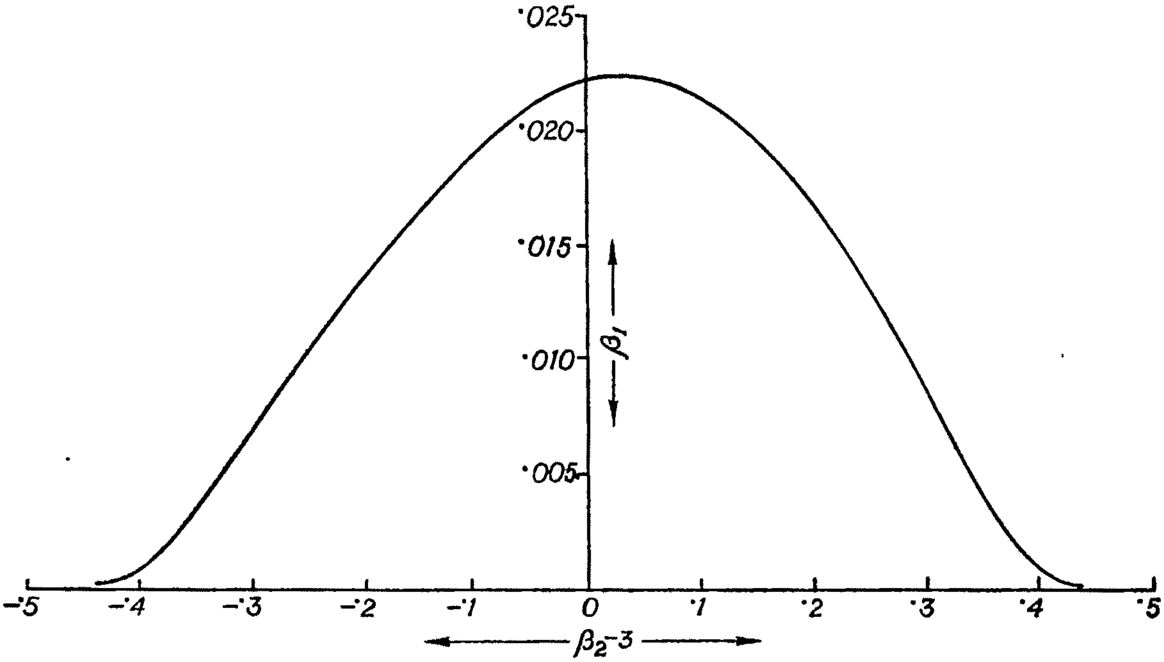

Table 6. Values of Hermite Polynomials for negative values of x.

x	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7
—2	1	—2	3	—2	—5	18	—11	—86
—1	1	—1	0	2	—2	—6	16	20
—·9	1	—·9	—·19	1·971	—1·2039	—6·80049	12.139941	29.876993
—·8	1	—·8	—·26	1 888	—0·4304	—7·20768	7·918144	36·911565
—·7	1	—·7	—·51	1·757	0·3001	—7·23807	3·566149	40·932116
—·6	1	—·6	—·64	1·584	0 9696	—6·91776	— 0·697344	41·924966
— 5	1	—·5	—·75	1 375	1·5625	—6·28125	— 4·671875	40·023438
—·4	1	—·4	—·84	1 136	2·0656	—5·37024	— 8·179904	35·493402
—·3	1	—·3	—·91	·873	2·4681	—4·23243	—11·070771	28·715811
—·2	1	—·2	— 96	·592	2·7616	—2 92032	—13·223936	20·166707
—·1	1	—·1	— 99	·299	2·9401	—1·49001	—14·551499	10·395210
— ·09	1	— 09	—·9919	269271	2·951466	—1 342716	—14·636484	9·378579
—·08	1	— 08	—·9936	·239488	2 961641	—1·194883	—14·712614	8·846809
— 07	1	—·07	— 9951	·209657	2·970624	—1·046572	—14·779860	7·814020
—·06	1	— 06	—·9964	179784	2·978413	— 897841	—14·838194	6.277836
—·05	1	—·05	—·9975	·149875	2 985006	— ·748750	—14·887594	5·236882
—·04	1	—·04	—·9984	·119936	2·990403	— ·599860	—14·928038	3·596161
—·03	1	—·03	—·9991	·089973	2·994601	— ·449730	—14·959512	3·147165
—·02	1	—·02	—·9996	·059992	2·997600	— ·299920	—14 982002	2·090160
—·01	1	—·01	—·9999	·029999	2·999400	— ·149990	—14·995500	2·999100
0	1	—0	—1	0	3	0	—15	0

Table 7. Values of Hermite Polynomials for positive values of x.

x	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7
0	1	0	—1	0	3	0	—15	0
·01	1	·01	—·9999	—·029999	2·999400	·149990	—14·995500	— 2·999100
·02	1	·02	—·9996	—·059992	2·997600	·299920	—14·982002	— 2·099160
·03	1	·03	—·9991	—·089973	2·994601	·449730	—14·959512	— 3·147165
·04	1	·04	—·9984	—·119936	2·990403	·599360	—14·928038	— 3·596161
·05	1	·05	—·9975	—·149875	2·985006	·748750	—14·887594	— 5·236882
·06	1	·06	—·9964	—·179784	2·978413	·897841	—14·838194	— 6·277336
·07	1	·07	—·9951	—·209657	2.970624	1·046572	—14·779860	— 7·314020
·08	1	·08	—·9936	—·239488	2·961641	1·194883	—14·712614	— 8·346309
·09	1	·09	—·9919	—·269271	2 951466	1·342716	—14·636484	— 9·373579
·1	1	·1	—·99	— ·299	2·9401	1·49001	—14·551499	—10·395210
·2	1	·2	—·96	— ·592	2·7616	2·92032	—13·223936	—20·166707
·3	1	·3	—·91	— ·873	2·4681	4·23243	—11·070771	—28·715811
·4	1	·4	—·84	— ·136	2·0656	5·37024	— 8·179904	—35·493402
·5	1	·5	—·75	—1·375	1·5625	6·28125	— 4·671875	—40·023438
·6	1	·6	—·64	—1·584	·9696	6·91776	— ·697344	—41·924966
·7	1	·7	—·51	—1·757	·3001	7·23807	3 566149	—40·932116
·8	1	·8	—·36	—1·888	— ·4304	7·20768	7·918144	—36·911565
·9	1	·9	—·19	—1·971	—1·2039	6·80049	12.139941	—29·876993
1	1	1	0	—2	—2	6	16	—20
2	1	2	3	2	—5	—18	—11	86

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ON THE LATTICE OF SUBGROUPS OF FINITE GROUPS

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1. Our object is to determine the lattices of subgroups of certain groups The meet of two subgroups is a subgroup and consequently if one takes this meet of two subgroups as the lattice product, the subgroups of a given group form a semilattice. If in this semilattice, one takes the so called free product, that is, the smallest subgroup containing the join of the two subgroups, as the lattice sum, then the above semilattice becomes a lattice. An easy example is given by the subgroups of the four group G which is generated by two elements a, b with the relations $a^2 = b^2 = I$. $ab = ba$. Here the subgroups are I, A, B, C where A, B and C are generated by a, b, and $c = ab$ respectively. The lattice of these subgroups is the well known birectangular lattice which has the Hasse diagram formed by two rectangles having three common vertices inclusive of G and I. Conversely, if a group of order 4 has this lattice as the lattice of its subgroups, then the subgroups A, B, C are groups of order 2 and hence it will be a four group. Indeed, if A, B, C are generated by a, b, c respectively, then $c = ab = ba$; for otherwise the number of elements of G will be more than four.

In a similar manner, the lattices of subgroups of octic group, the dihedral group of order 2^n and the generalised quaternion group of order 2^n have been determined. Further, it has been shown that the subgroups of a cyclic group C_n of finite order n form a lattice which is isomorphic to the lattice of the factors of n and conversely, if the lattice of subgroups of any finite group be isomorphic to the lattice of the factors of the order of the group, then that group is cyclic. This generalises the result proved by R. Baer that $L(G)$ is a chain if and only if G is a cyclic group of prime power order. Moreover, a complete relation between the lattice of subgroups of a group G and the lattice of subgroups of the factor group G/N, corresponding to any normal subgroup N of G is established.

2. Let C_n be a cyclic group of order n, say, generated by an element a, so that $a^n = I$. Then if G is any subgroup of C_n, then G is also a cyclic group of order k, where k is a factor of n. Conversely, for each factor k of n, there exists a cyclic subgroup G of C_n. Thus the subgroups of C_n are in biunique correspondence with the factors of n. This correspondence can be proved to be an isomorphism between the lattice of subgroups of C_n and the lattice of the factors of n.

Indeed if we take $m \supseteq r$ when r is a divisor of m, then it is seen that $C_m \supseteq C_r$ according as $m \supseteq r$. But with the inclusion $m \supseteq r$ when r is a divisor of m, the factors of n form a lattice in which the lattice sum is the l. c. m $[a, b]$ and the lattice product is the h. c. f (a, b), the factors being determined except for unities. The correspondence $C_m \longleftrightarrow m$ is therefore an isomorphism between the two lattices.

Conversely, let us suppose that $L(C_n) \simeq L(n)$, where $L(C_n)$ is the lattice of subgroups of any finite group C_n of order n and $L(n)$ is the lattice of the factors of n and $n = p_1^{r_1} \cdot p_2^{r_2} \dots p_m^{r_m}$. We shall prove that C_n is cyclic.

As $L(n)$ is distributive, $L(C_n)$ is also so.

As $L(C_n)$ is distributive, C_n is a generalised cyclic group. [O. Ore. (1938)]

i.e., if a, $b \in C_n$, then $a = c^m$, $b = c^n$, where $c \in C_n$.

Hence, $ab = c^m \cdot c^n = c^{m+n} = c^{n+m} = c^n \cdot c^m = ba$.

Therefore C_n is Abelian.

Hence by the basis theorem

$$C_n = C_{q_1^{s_1}} \times C_{q_2^{s_2}} \times \dots \times C_{q_t^{s_t}},$$

where $C_{q_i^{s_i}}$ are cyclic groups of prime power orders $q_i^{s_i}$.

As $C_{q_i^{s_i}}$ is a subgroup of C_n, its order $q_i^{s_i}$ is a factor of n. As n has unique factorisation, we have $q_i^{s_i} = p_j^{r_j}$, say.

$$\therefore \qquad C_n = C_{p_1^{r_1}} \times C_{p_2^{r_2}} \times \dots \times C_{p_m^{r_m}},$$

where p_i's are distinct primes.

Let $c_1, c_2, \dots, c_m$ be the generators of $C_{p_1^{r_1}}, C_{p_2^{r_2}}, \dots, C_{p_m^{r_m}}$ respectively.

That is, $(c_i)^{p_i^{r_i}} = I$, $i = 1, 2, \dots, m$.

$$\therefore \qquad (\Pi c_i)^{\Pi p_i^{r_i}} = \Pi (c_i)^{\Pi p_i^{r_i}} = \Pi\left(c_i^{p_i^{r_i}}\right)^{\prod_{j=1}^{m} p_j^{r_j} (j \text{ not equal to } i)} = I$$

Hence $b = c_1 c_2 \dots c_m$ will generate a cyclic group whose order will be the l. c. m. of the orders of $C_{p_1^{r_1}}, C_{p_2^{r_2}}, \dots, C_{p_m^{r_m}}$; that is, equal to n, as p_i's are distinct primes.

As $L(C_n) \simeq L(n)$, i.e., as $n \longleftrightarrow C_n$; C_n is identical with this cyclic group $\{b\}$. Thus $L(C_n) \simeq L(n)$ if and only if C_n is cyclic.

As an application, one can consider the lattice of subfields of a Galois field GF_{p^n}. The non-zero elements of this Galois field form a cyclic group of order $p^n - 1$. Subgroups of this cyclic group form a lattice isomorphic to the lattice of the factors of $p^n - 1$.

To each divisor m of n, there exists exactly one subfield GF_{p^m} of GF_{p^n}.

Again if $p^m - 1$ is a factor of $p^n - 1$, then the cyclic group of order $p^m - 1$ is a subgroup of the cyclic group of order $p^n - 1$. Let us now include the zero element in both the cyclic groups. Then the two cyclic groups will have p^m and p^n elements respectively. But as p^m and p^n determine respectively GF_{p^m} and GF_{p^n} uniquely, we have $GF_{p^m} \subset GF_{p^n}$, i.e., m is a factor of n. And as every cyclic subgroup of the cyclic group is uniquely determined by its order it follows that to each cyclic subgroup $C_{(p^m-1)}$ of $C_{(p^n-1)}$ there is a subfield GF_{p^m} of the Galois field GF_{p^n}.

Conversely, if m is a factor of n, then $GF_{p^m} \subset GF_{p^n}$. The non-zero elements of GF_{p^m} and GF_{p^n} will form respectively the cyclic groups of order p^m-1 and p^n-1 and $C_{(p^m-1)} \subset C_{(p^n-1)}$. Thus p^m-1 is a factor of p^n-1 and to each sub-field of the Galois field, there is a cyclic subgroup.

Hence from all these we have :
lattice of sub-fields of a Galois field $GF_{p^n} \simeq$ lattice of subgroups of the cyclic group of order p^n-1

$\simeq$ lattice of the factors of p^n-1

$\simeq$ lattice of the factors of n

3. The octic group is generated by two elements a, b with the relations $a^4 = b^2 = I$, $ba = a^3b$

The subgroups of this group are given by the followings :—

G_4^1 : I, a, a^2, a^3 ; $a^4 = I$

G_4^2 : I, a^2, b, a^2b

G_4^3 : I, a^2, ab, a^3b

G_2^1 : I, a^2

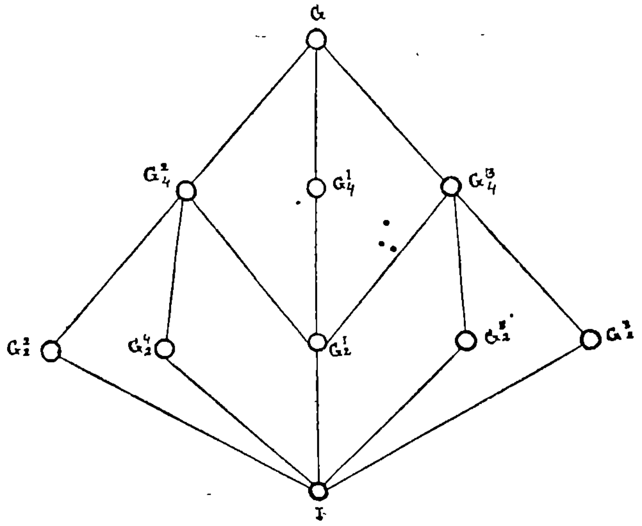
G_2^2 : I, b

G_2^3 : I, ab

G_2^4 : I, a^2b

G_2^5 : I, a^3b

The subgroup lattice is given by the following Hasse diagram :



Conversely, we shall prove that if G be a group of order 8 and the lattice of the subgroups of G be represented by the above Hasse diagram, then G is an octic group.

As G is of order 8, its proper subgroups will be of order 4, 2, 1. Hence the order G^i_2, $i = 1, \ldots, 5$, is 2 and that of G^j_4, $j = 1, 2, 3$ is 4.

Now $L(G^1_4)$ is a chain, therefore G^1_4 is a cyclic group of order 4, i.e., $a^4 = I$ [R. Baer, (1939)].

Therefore, $G^1_2 \cdot (a^2)^2 = I$.

Moreover, G^2_4, $G^3_4 \supset G^1_2$, hence I, $a^2 \in G^2_4$, G^3_4. Also a, a^3 not $\in G^2_4$, G^3_4 as G^1_2 is the only subgroup common to G^1_4, G^2_4 and G^3_4. The elements of G are I, a, a^2, a^3, b, ab, a^2b, a^3b; where $a^4 = I$.

Let us suppose that b is an element of G^2_4 The elements of G^2_4 are then I, a^2, b, a^2b. The elements of G^3_4 are therefore, I, a^2, ab, a^3b. Now, as G^2_4 and G^3_4 are of order 4 and $L(G^2_4)$ and $L(G^3_4)$ are birectangular lattices, G^2_4 and G^3_4 are four groups. Therefore, in G^2_4 we have $b^2 = I$ and in G^3_4 we have $(ab)^2 = I$ i.e., $ba = a^3b$. Hence G is generated by two elements a, b with the relations $a^4 = b^2 = I$, $ba = a^3b$. Therefore, G is an octic group.

Thus, a group G of order 8 is an octic group if and only if $L(G)$ is represented by the Hasse diagram given beforehand.

4. The dihedral group D_{2^n} of order 2^n is generated by two elements a, b with the relations

$$a^{2^{n-1}} = I,\ b^2 = I,\ bab^{-1} = a^{-1}; \quad n \geq 3$$

It has the following three subgroups of order 2^{n-1}:

(i) $a^{2^{n-1}} = I$

(ii) $(a^2)^{2^{n-2}} = I,\ b^2 = I,\ ba^2b^{-1} = (a^2)^{-1}$

(iii) $(a^2)^{2^{n-2}} = I,\ (ab)^2 = I,\ (ab)a^2(ab)^{-1} = (a^2)^{-1}$;

of which the first is cyclic and the other two are again dihedral groups having the cyclic group $(a^2)^{2^{n-2}} = I$ of order 2^{n-2} in common between them. Now (ii) and (iii) being again dihedral groups, each of them will have three subgroups of order 2^{n-2} of which one will be the cyclic group $(a^2)^{2^{n-2}} = I$ and the other two will be again dihedral groups and all these five groups of order 2^{n-2} will have the cyclic subgroup $(a^4)^{2^{n-3}} = I$ of order 2^{n-3} in common Therefore, there will be nine subgroups—one cyclic and eight dihedral—of order 2^{n-3}, and so on.

Proceeding in this way, we have ultimately $2^{n-1}+1$ subgroups of order 2 each of which will cover I, that is, there is no subgroup lying between them and I. Thus $L(D_{2^n})$ is obtained.

Conversely, if $L(D_{2^n})$ of a group D_{2^n} of order 2^n is given by the above structure, we shall prove that D_{2^n} is a dihedral group. To prove this we shall apply mathematical induction.

The theorem has been proved to be true for $n = 3$, in the case of an octic group.

Let us now suppose that the theorem is true for $n = m$. That is, if $L(D_{2^m})$ has the structure given above, then D_{2^m} is a dihedral group.

Let now $L(D_{2^{m+1}})$ be given by the above structure. $D_{2^{m+1}}$ has three subgroups $G^i_{2^m}$, $i = 1, 2, 3$; of order 2^m of which one subgroup, say $G^1_{2^m}$, is such that $L(G^1_{2^m})$ is a chain. Hence $G^1_{2^m}$ is a cyclic group of order 2^m, that is, $a^{2^m} = I$.

The only subgroup common to $G^1_{2^m}$, $G^2_{2^m}$, $G^3_{2^m}$ is evidently the cyclic subgroup $(a^2)^{2^{m-1}} = I$ of order 2^{m-1}.

Hence $$\{a^2\} \subset G^2_{2^m},\ G^3_{2^m}.$$

Hence $D_{2^{m+1}}$ is generated by two elements a, b with $a^{2^m} = I$. Let $b \in G^2_{2^m}$. Then $G^2_{2^m}$ is generated by two elements a^2, b with $(a^2)^{2^{m-1}} = I$ and hence $G^3_{2^m}$ is generated by a^2, ab with $(a^2)^{2^{m-1}} = I$.

Now $L(G^2_{2^m})$ and $L(G^3_{2^m})$ are represented by the structure given beforehand. Hence by the induction hypothesis, they are dihedral groups.

Therefore, in $G^2_{2^m}$, we have $b^2 = I$ and in $G^3_{2^m}$, we have $(ab)^2 = I$, that is, $bab^{-1} = a^{-1}$.

Thus $D_{2^{m+1}}$ is generated by two elements a, b with the relations $a^{2^m} = I$, $b^2 = I$, $bab^{-1} = a^{-1}$. Hence $D_{2^{m+1}}$ is a dihedral group of order 2^{m+1}.

Thus if the theorem is true for $n = m$, it is also true for $n = m+1$. Hence by mathematical induction, the theorem is true always.

Thus, a group D_{2^n} of order 2^n is dihedral if and only if $L(D_{2^n})$ is represented by the structure given beforehand.

5. Generalised quaternion group G_{2^n} of order 2^n is generated by two elements a, b with the relations $a^{2^{n-1}} = I$, $b^2 = a^{2^{n-2}}$, $bab^{-1} = a^{-1}$; $n \geq 3$

The subgroups of order 2^{n-1} of this generalised quaternion group are the followings:

(i) $a^{2^{n-1}} = I$

(ii) $(a^2)^{2^{n-2}} = I$, $b^2 = (a^2)^{2^{n-3}}$, $ba^2b^{-1} = (a^2)^{-1}$

(iii) $(a^2)^{2^{n-2}} = I$, $(ab)^2 = (a^2)^{2^{n-3}}$, $(ab)a^2(ab)^{-1} = (a^2)^{-1}$

of which the first is cyclic and the other two are again generalised quaternion having the cyclic group $(a^2)^{2^{n-2}} = I$ of order 2^{n-2} in common between them. Now (*ii*) and (*iii*) being again generalised quaternion groups, each of them will have three subgroups of order 2^{n-2} of which one will be the cyclic group $(a^2)^{2^{n-2}} = I$ and the other two will be again generalised quaternion groups and all these five groups of order 2^{n-2} will have the cyclic subgroup $(a^4)^{2^{n-3}} = I$ of order 2^{n-3} in common. Therefore, there will be nine subgroups —one cyclic and eight generalised quaternion—of order 2^{n-3}; and so on. And as we know that a generalised quaternion group will contain only one subgroup of order 2, all the $2^{n-2}+1$ subgroups of order 2^2 will cover only one subgroup $(a^{2^{n-2}})^2 = I$ of order 2 which will again cover I. The quaternion groups are groups of order 2^3. There are $2^{n-3}+1$ subgroups of order 2^3 of which one is cyclic and 2^{n-3} subgroups are quaternion.

The elements $a^{2^{n-3}}$, a^ib ; $i = 0,\ 1,\ 2,\ \ldots,\ 2^{n-3}-1$; with the relations

$$(a^{2^{n-3}})^{2^2} = I,\quad (a^ib)^2 = (a^{2^{n-3}})^2,$$

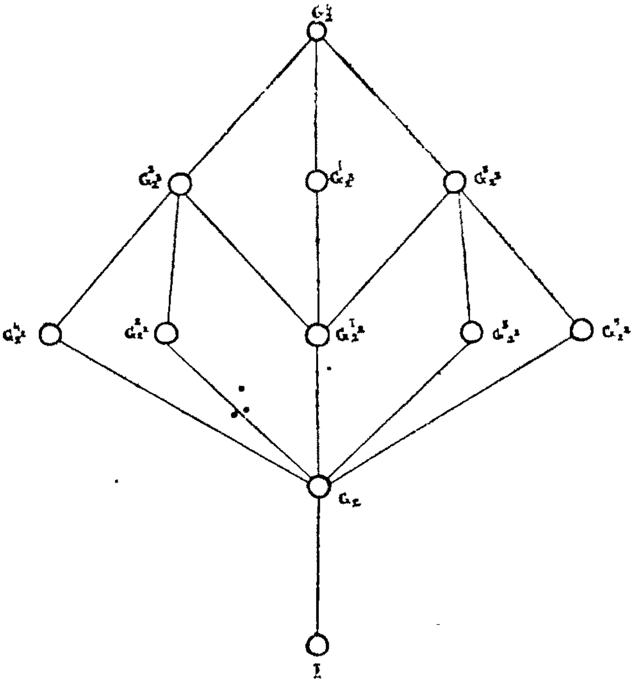
$$(a^ib)a^{2^{n-3}}(a^ib)^{-1} = (a^{2^{n-3}})^{-1}$$

will generate the 2^{n-3} quaternion groups

Conversely, if $L(G_{2^n})$ is given by the above structure, G_{2^n} has only one subgroup of order 2 and hence G_{2^n} is either a cyclic group or a generalised quaternion group. But it can not be cyclic, for in that case $L(G_{2^n})$ will be a chain. Hence G_{2^n} is a generalised quaternion group.

Hence, a group G_{2^n} of order 2^n is generalised quaternion if and only if $L(G_{2^n})$ has the structure given above.

The Hasse diagram of a quaternion group of order 2^4 is the following:



6. Let G be a group and N any subgroup of G. Then all subgroups G_r of G which includes N will form a lattice. For, if $N \subseteq G_i$, $G_j \subseteq G$; then $N \subseteq G_i + G_j \subseteq G$ and $N \subseteq G_i \cap G_j \subseteq G$.

Now let N be a normal subgroup of G. Then G/N is a factor group.

If G_r be the subgroups of G such that $N \subseteq G_r \subseteq G$, then N is also a normal subgroup of G_r.

Thus the factorgroups G_r/N exist and $G_r/N \subseteq G/N$.

On the other hand, if B be a subgroup of the factorgroup G/N, then $B = G'/N$, where G' is the join of those cosets of N in G which form the subgroup B.

Thus the subgroups of a factorgroup are factorgroups. Combining the two, we conclude that the subgroups of G/N are G_r/N where $N \subseteq G_r \subseteq G$.

The subgroups of G/N will form a lattice $L(G/N)$ with G/N as the greatest and $N/N = I$ as the least elements

Again, the subgroups G_r for which $N \subseteq G_r \subseteq G$, form a lattice $L(G_N)$, N being in this case a normal subgroup of G.

Now all the subgroups G_r, where $N \subseteq G_r \subseteq G$, correspond to all the subgroups G_r/N of G/N.

Let us now set up the correspondence $G_r \rightarrow G_r/N$ between the elements of $L(G_N)$ and $L(G/N)$. Let $G_r \rightarrow G_r/N$, $G_s \rightarrow G_s/N$. If $G_r \supseteq G_s$, then $G_r/N \supseteq G_s/N$; and conversely.

Hence $L(G_N) \simeq L(G/N)$.

Again, the subgroups G'_r, for which $I \subseteq G'_r \subset N$ will be annihilated in the lattice $L(G/N)$.

Now, let $\bar{G}$ be a subgroup of G which neither includes N nor is included in it.

As G is homomorphic to G/N, the subgroup $\bar{G}$ of G will be homomorphic to a subgroup A' of G/N.

i.e.,

$$\bar{G} \sim A' \subseteq G/N$$

The nucleus of this homomorphism is $\bar{G} \cap N \doteq G_0$, say.

Hence by the theorem of isomorphism, we have

$$A' \simeq \bar{G}/G_0$$

Again, as $\bar{G}$ is a subgroup and N is a normal subgroup of G, we have, by the first theorem of isomorphism

$$\bar{G}/G_0 \simeq \frac{\bar{G} \cdot N}{N} = G'/N,$$

where G' is the smallest subgroup of G containing both $\bar{G}$ and N.

Hence all the subgroups $\bar{G}$ of G which neither include N nor are included in it correspond to G'/N, G' being the least subgroup of G containing both $\bar{G}$ and N.

Thus (*i*) $L(G_N) \simeq L(G/N)$

(*ii*) The subgroups G'_r of G, for which $I \subseteq G'_r \subset N$, are annihilated in the lattice $L(G/N)$.

(*iii*) If $\bar{G}$ be a subgroup of G which neither includes N nor is included in it then $\bar{G} \sim G'/N$ where G' is the smallest subgroup of G containing both $\bar{G}$ and N;

establishing a complete relation between the lattice $L(G)$ of the subgroups of a given group G and the lattice $L(G/N)$ of the subgroups of the factorgroup G/N, corresponding to any normal subgroup N of G.

In fine, I take the opportunity of thanking Dr. A. C. Choudhury for his guidance during the work.

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SPECIAL METRIC FORMS AND THEIR GRAVITATIONAL SIGNIFICANCE

By

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Abstract. Four special cases of a metric are discussed so as to find out the gravitational situations described by them. In the solutions obtained, wherever possible, the full complement of the arbitrary functions is stated distinguishing them from certain well-known cases. The general background of the investigation is explained in the introduction and in the last section.

1. Introduction. In the course of a recent investigation on line-elements of the form

$$ds^2 = -A\,dx^2 - B\,dy^2 - C\,dz^2 + D\,dt^2, \qquad (1{\cdot}1)$$

where A, B, C, D are functions of x, y, z and t, we came across some interesting results for a few special cases of (1·1), which we consider worth being placed on record.

The special cases considered by us are the following

(*i*)
$$ds^2 = -\theta\,dx^2 - dy^2 - dz^2 + dt^2,$$
$$\theta = \theta(x, y, z, t)\,; \qquad (1{\cdot}2)$$

(*ii*)
$$ds^2 = -dx^2 - dy^2 - dz^2 + \psi\,dt^2,$$
$$\psi = \psi(x, y, z, t)\,; \qquad (1{\cdot}3)$$

(*iii*)
$$ds^2 = -\varphi(dx^2 + dy^2 + dz^2) + dt^2,$$
$$\varphi = \varphi(x, y, z, t)\,; \qquad (1{\cdot}4)$$

(*iv*)
$$ds^2 = \varphi(-dx^2 - dy^2 - dz^2 + dt^2),$$
$$\varphi = \varphi(x, y, z, t)\,; \qquad (1{\cdot}5)$$

So far no physical significance has been found of the mathematical conditions which are responsible for a line-element in general to be transformable into the form (1·1). Eisenhart (1926) has given the mathematical conditions and it is also well-known that for the form (1·1)

$$R_{hijk} = 0, \qquad (h, i, j, k \neq), \qquad (1{\cdot}6)$$

in the usual notation. In general a Gaussian transformation can be suitably chosen to fix four out of the ten metric potentials. But (1·1) implies that two more are fixed. On account of the identity

$$R_{hijk} + R_{hjki} + R_{hkij} = 0, \qquad (h, i, j, k \neq), \qquad (1{\cdot}7)$$

satisfied by the Riemann-Christoffel tensor, (1·6) implies two conditions. But they are not tensor conditions.

The particular cases considered by us are all distinguished by the simplifying condition that only one variable potential is present in the line-element. The gravitational field equations

$$R_{ij} = 0,$$

being ten in number, the one potential function is subjected to far too many conditions. The four identities reduce the restrictions only to a certain extent. One does not therefore anticipate a non-trivial solution. We have examined what happens in each case assuming the field equations to be

$$\begin{aligned} &(a) \qquad R_{\mu\nu} = 0, \\ &(b) \qquad R_{\mu\nu} = \lambda g_{\mu\nu}, \\ &(c) \qquad R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} + \lambda g_{\mu\nu} = -\kappa T_{\mu\nu}, \end{aligned} \tag{1·8}$$

where κ stands for 8π in natural units. It will be assumed that

$$T_{\mu\nu} = (p+\varrho) v_\mu v_\nu - p g_{\mu\nu}, \tag{1·9}$$

p and ϱ being the pressure and density of the distribution and v_μ the flow vector satisfying

$$g^{\mu\nu} v_\mu v_\nu = 1. \tag{1·10}$$

The object of the investigation is to find if any simple non-trivial solution exists of the field equations characterized by the existence of only one variable potential. Cases (*iii*) and (*iv*) have already attracted some attention in cosmological discussions and in connection with Milne's Kinematical theory. Our actual findings in the four cases are as reported below.

2. Case (i): Metric (1·2). The components of the curvature tensor are given by

$$\begin{aligned} R_{1\mu 1\nu} &= \tfrac{1}{2}\theta_{\mu\nu} - \frac{1}{4\theta}\theta_\mu \theta_\nu, && (\mu, \nu = 2, 3, 4), \\ R_{\lambda\mu\nu 1} &= 0, && (\lambda, \mu, \nu = 2, 3, 4); \\ R_{\lambda\mu\nu\sigma} &= 0, && (\lambda, \mu, \nu, \sigma = 2, 3, 4); \end{aligned} \tag{2·1}$$

where

$$\theta_\mu \equiv \frac{\partial \theta}{\partial x^\mu}; \quad \theta_{\mu\nu} \equiv \frac{\partial^2 \theta}{\partial x_\mu \partial x_\nu};$$

and

$$x^1 \equiv x, \quad x^2 \equiv y, \quad x^3 \equiv z, \quad x^4 \equiv t.$$

The components of the contracted curvature tensor are given by

$$R_{11} = \tfrac{1}{2}(\theta_{22} + \theta_{33} - \theta_{44}) - \frac{1}{4\theta}(\theta_2^2 + \theta_3^2 - \theta_4^2);$$

$$R_{1\mu} = 0, \qquad (\mu = 2, 3, 4);$$

$$R_{\mu\nu} = \frac{1}{2\theta}\theta_{\mu\nu} - \frac{1}{4\theta^2}\theta_\mu\theta_\nu, \qquad (\mu, \nu, = 2, 3, 4); \qquad (2{\cdot}2)$$

which satisfy the relation

$$R_{11} = \theta(R_{22}+R_{33}-R_{44}). \qquad (2{\cdot}3)$$

Now an inspection of (2·1) and (2·2) shows that when the field equations (1·8a) are satisfied we have a flat space-time. From (2·3) it is clear that (1·8b) cannot be satisfied for $\lambda \neq 0$. Thus we are led to the conclusion that there is no gravitational field in empty space of the type (1·2), that is, of a type which implies a preferential metrical behaviour in one spatial direction.

Subject to the field equations (1·8c), the metric gives an interesting solution corresponding to no flow of matter, v_1 being zero by virtue of the field equations themselves.. We find in this case

$$\theta^{\frac{1}{2}} = A(x)(y^2+z^2+t^2)+B_1(x)y+B_2(x)z+B_3(x)t+C(x), \qquad (2{\cdot}4)$$

$$\kappa p = \lambda \qquad (2{\cdot}5)$$

$$\kappa\varrho = -\frac{4A}{\theta^{\frac{1}{2}}} - \lambda, \qquad (2{\cdot}6)$$

where A, B_1, B_2, B_3 and C are all arbitrary functions of x. This solution requires λ to be positive. It further puts a severe restriction on the behaviour of A, as ϱ has to be positive.

3. **Case (ii): Metric (3).** Here the components of the curvature tensor are given by

$$R_{\mu4\nu4} = -\tfrac{1}{2}\psi_{\mu\nu} + \frac{1}{4\psi}\psi_\mu\psi_\nu, \qquad (\mu, \nu = 1, 2, 3);$$

$$R_{\lambda\mu\nu4} = 0, \qquad (\lambda, \mu, \nu = 1, 2, 3);$$

$$R_{\lambda\mu\nu\sigma} = 0, \qquad (\lambda, \mu, \nu, \sigma = 1, 2, 3); \qquad (3{\cdot}1)$$

The components of the contracted curvature tensor are found to be

$$R_{\mu\nu} = \frac{1}{2\psi}\psi_{\mu\nu} - \frac{1}{4\psi^2}\psi_\mu\psi_\nu, \qquad (\mu, \nu = 1, 2, 3);$$

$$R_{\mu4} = 0, \qquad (\mu = 1, 2, 3); \qquad (3{\cdot}2)$$

$$R_{44} = -\tfrac{1}{2}(\psi_{11}+\psi_{22}+\psi_{33}) + \frac{1}{4\psi}(\psi_1^2+\psi_2^2+\psi_3^2);$$

where the following relation exists between them analogous to that in the previous case:

$$R_{44} = -\psi(R_{11}+R_{22}+R_{33}) \qquad (3{\cdot}3)$$

Here again we find that (1·8a) implies

$$R_{\lambda\mu\nu\sigma} = 0,$$

giving flat space-time. There is no solution satisfying (1·8b).

For the equations (1·8c) the only solution available corresponds to

$$v_1 = v_2 = v_3 = 0.$$

The values of ψ, p and ϱ are as follows:

$$\psi^{\frac{1}{4}} = A(t)(x^2+y^2+z^2)+B_1(t)x+B_2(t)y+B_3(t)z+C(t) \tag{3·4}$$

$$\kappa p = \frac{4A}{\psi^{\frac{1}{4}}}+\lambda, \tag{3·5}$$

$$\kappa\varrho = -\lambda, \tag{3·6}$$

where A, B_1, B_2, B_3 and C are arbitrary functions of t. Thus we have found the most general form (3·4) for ψ, for which the perfect fluid distribution holds good. Here λ must be negative and as p must be positive there is a corresponding restriction on A.

It is interesting, however, to observe that for the metric (1·3)

$$R = 0$$

means

$$\nabla^2\psi^{\frac{1}{4}} = 0.$$

This condition in the case of a perfect fluid distribution cannot be satified as this would mean

$$A = 0.$$

4. Case (iii): Metric (1·4). For this metric we find the componetns of the curvature tensor to be:

$$\begin{aligned}
R_{\lambda\mu\lambda\mu} &= \tfrac{1}{2}(\varphi_{\lambda\lambda}+\varphi_{\mu\mu})-\frac{1}{4\varphi}(2\varphi_\lambda^2+2\varphi_\mu^2-\varphi_\nu^2)-\frac{1}{4}\varphi_4^2,\\
R_{\lambda\mu\lambda\nu} &= \tfrac{1}{2}\varphi_{\mu\nu}-\frac{3}{4\varphi}\varphi_\mu\varphi_\nu,\\
R_{\lambda\mu\lambda4} &= \tfrac{1}{2}\varphi_{\mu4}-\frac{1}{2\varphi}\varphi_\mu\varphi_4,\\
R_{\lambda4\lambda4} &= \tfrac{1}{2}\varphi_{44}-\frac{1}{4\varphi}\varphi_4^2,\\
R_{\lambda4\mu4} &= 0,\\
R_{\lambda\mu\nu4} &= 0;
\end{aligned} \tag{4·1}$$

and the components of the contracted curvature tensor to be:

$$
\begin{aligned}
R_{\mu\nu} &= \frac{1}{2\varphi}\varphi_{\mu\nu} - \frac{3}{4\varphi^2}\varphi_\mu\varphi_\nu, \\
R_{\mu 4} &= \frac{1}{\varphi}\varphi_{\mu 4} - \frac{1}{\varphi^2}\varphi_\mu\varphi_4, \\
R_{44} &= \frac{3}{2\varphi}\varphi_{44} - \frac{3}{4\varphi^2}\varphi_4^{\,2}, \qquad\qquad (4{\cdot}2) \\
R_{\mu\mu} &= \frac{1}{2\varphi}(2\varphi_{\mu\mu} + \varphi_{\lambda\lambda} + \varphi_{\nu\nu}) - \tfrac{1}{2}\varphi_{44} - \frac{1}{4\varphi}\varphi_4^{\,2} - \frac{1}{4\varphi^2}(4\varphi_\mu^{\,2} + \varphi_\lambda^{\,2} + \varphi_\nu^{\,2});
\end{aligned}
$$

where λ, μ, ν are definite and unequal suffixes, which run over 1, 2, 3 only.

A close examination of (4·1) and (4·2) reveals that here again the field equation (1·8a) means a flat space-time. Equation (1·8b), however, does not lead to incompatibility as in the previous two cases. We find that when (1·8b) is satisfied, all the curvature components except those of the form

$$R_{\lambda\mu\lambda\mu}, \qquad (\lambda, \mu = 1, 2, 3, 4),$$

vanish and

$$
\begin{aligned}
R_{1212} = R_{1313} = R_{2323} &= -(1/3)\varphi^2\lambda, \\
R_{1414} = R_{2424} = R_{3434} &= (1/3)\varphi\lambda.
\end{aligned}
$$

We find the most general solution of (1·8b) to be

$$\varphi = \left[\frac{f(t)}{A(x^2+y^2+z^2)+B_1x+B_2y+B_3z+C}\right]^2, \qquad (4{\cdot}3)$$

where A, B_1, B_2, B_3 and C are arbitrary constants and f is a function of t alone satisfying

$$\dot{f}^2 = 1/3\lambda f^2 + \alpha, \qquad \alpha = -4AC + B_1^{\,2} + B_2^{\,2} + B_3^{\,2}. \qquad (4{\cdot}4)$$

In the above an overhead dot denotes a differentiation with respect to t. Integrating (4·4), we have the following alternative forms for f:

$$f = \sqrt{\frac{3\alpha}{\lambda}}\sinh\sqrt{\frac{\lambda}{3}}(t-t_0), \qquad \lambda > 0,\ \alpha > 0; \qquad (4{\cdot}5a)$$

$$f = \sqrt{\frac{3\alpha}{-\lambda}}\cosh\sqrt{\frac{\lambda}{3}}(t-t_0), \qquad \lambda > 0,\ \alpha < 0; \qquad (4{\cdot}5b)$$

$$f = \exp\left[\pm\sqrt{\frac{\lambda}{3}}(t-t_0)\right], \qquad \lambda > 0; \qquad (4{\cdot}5c)$$

$$f=\sqrt{-\frac{3\alpha}{\lambda}}\cos\sqrt{-\frac{\lambda}{3}}\,(t-t_0),\qquad \lambda<0,\ \alpha>0. \tag{4·5d}$$

Now the solution for φ given by (4·3) and (4·5) can be put, after some suitable elementary transformations, in some standard forms. The elementary transformations used are shift of the origin of space, change of length in all directions and rotation of the space axes. The forms thus obtained are, when $A\neq0$,

$$\varphi=\left[\frac{(1/R_0)\sqrt{(3/\lambda)}\sinh(\sqrt{\lambda/3})(t-t_0)}{1-r^2/4R_0^2}\right]^2,\qquad \lambda>0; \tag{4·6a}$$

$$\varphi=\left[\frac{(1/R_0)\sqrt{(3/\lambda)}\cosh(\sqrt{\lambda/3})(t-t_0)}{1+r^2/4R_0^2}\right]^2,\qquad \lambda>0; \tag{4·6b}$$

$$\varphi=[(1/r^2)\exp\{\pm\sqrt{(\lambda/3)}(t-t_0)\}]^2,\qquad \lambda>0; \tag{4·6c}$$

$$\varphi=\left[\frac{(1/R_0)\sqrt{(-3/\lambda)}\cos\sqrt{(-\lambda/3)}(t-t_0)}{1-r^2/4R_0^2}\right]^2,\qquad \lambda<0; \tag{4·6d}$$

and those obtained by taking $A=0$ are:

$$\varphi=\exp[\pm2\sqrt{(\lambda/3)}(t-t_0)]\qquad \lambda>0; \tag{4·7}$$

$$\varphi=\frac{3}{\lambda}\left[\frac{1}{x}\sinh\sqrt{\frac{\lambda}{3}}\,(t-t_0)\right]^2,\qquad \lambda>0; \tag{4·8a}$$

$$\varphi=-\frac{3}{\lambda}\left[\frac{1}{z}\cos\sqrt{-\frac{\lambda}{3}}\,(t-t_0)\right]^2,\qquad \lambda<0; \tag{4·8b}$$

the case $A=0$ being impossible for (4·5b). The forms (4·6) and (4·7) for φ are seen to be spatially isotropic, whereas those of (4·8) cannot be made so by any transformations.

The only solution of some interest is apparently de Sitter's, *viz.* (4·6b) and (4·7). However, (4·6c), (4·8a) and (4·8b) may also provide explanations of some gravitational situations of astronomical interest outside the domain of singularity. The forms (4·6a) and (4·6b) deserve attention.

We also find that there is no solution of Einstein's equations (1·8a) or (1·8b) so that

$$\varphi=\varphi(x,y,z).$$

Equation (1·8c) gives the following solution in this case

$$\varphi=\left[\frac{f(t)}{A(x^2+y^2+z^2)+B_1x+B_2y+B_3z+C}\right]^2, \tag{4·9}$$

p and ϱ being given by

$$\lambda-\kappa p=-\frac{\alpha}{f^2}+\frac{2\ddot{f}}{f}+\frac{\dot{f}^2}{f^2}, \tag{4·10}$$

$$\lambda+\kappa\varrho = -\frac{3\alpha}{f^2}+\frac{3\dot{f}^2}{f^2}, \tag{4·11}$$

where $$\alpha = -4AC+B_1^2+B_2^2+B_3^2,$$

and A, B_1, B_2, B_3 and C are constants.

Lemaitre's well-known form is a particular case of the above solution. One is free in the solution given here to put some of the arbitraries as zeros. In an astronomical solution where the outer field has to be of the form (1 4), the form obtained by us gives more latitude to fit the external solution to an internal solution.

5. Case (iv): Metric (1·5) We now consider a metric which is conformal to flat space-time. In this case the components of the curvature tensor are:

$$\begin{aligned}
R_{\mu\nu\mu\nu} &= \tfrac{1}{2}(\varphi_{\mu\mu}+\varphi_{\nu\nu})-\frac{1}{4\varphi}(2\varphi_\mu^2+\varphi_\nu^2-\varphi_\lambda^2)-\frac{1}{4\varphi}\varphi_4^2,\\
R_{\lambda\mu\lambda\nu} &= \tfrac{1}{2}\varphi_{\mu\nu}-\frac{3}{4\varphi}\varphi_\mu\varphi_\nu,\\
R_{\lambda\mu\lambda 4} &= \tfrac{1}{2}\varphi_{\mu 4}-\frac{3}{4\varphi}\varphi_\mu\varphi_4,\\
R_{\mu 4\nu 4} &= -\tfrac{1}{2}\varphi_{\mu\nu}+\frac{3}{4\varphi}\varphi_\mu\varphi_\nu,\\
R_{\lambda 4\lambda 4} &= -\tfrac{1}{2}(\varphi_{\lambda\lambda}-\varphi_{44})-\frac{1}{2\varphi}\varphi_4^2-\frac{1}{4\varphi}(-2\varphi_\lambda^2+\varphi_\mu^2+\varphi_\nu^2),\\
R_{\lambda\mu\nu 4} &= 0;
\end{aligned} \tag{5·1}$$

and those of the contracted curvature tensor are:

$$\begin{aligned}
R_{\lambda\lambda} &= \frac{1}{\varphi}\varphi_{\lambda\lambda}-\frac{3}{2\varphi^2}\varphi_\lambda^2+\frac{1}{2\varphi}\Box\varphi,\\
R_{44} &= \frac{1}{\varphi}\varphi_{44}-\frac{3}{2\varphi^2}\varphi_4^2-\frac{1}{2\varphi}\Box\varphi,\\
R_{\mu\nu} &= \frac{1}{\varphi}\varphi_{\mu\nu}-\frac{3}{2\varphi^2}\varphi_\mu\varphi_\nu,\\
R_{\lambda 4} &= \frac{1}{\varphi}\varphi_{\lambda 4}-\frac{3}{2\varphi^2}\varphi_\lambda\varphi_4;
\end{aligned} \tag{5·2}$$

where $$\Box\varphi = \varphi_{11}+\varphi_{22}+\varphi_{33}-\varphi_{44},$$

and where λ, μ, ν are definite and unequal suffixes, each taking the values 1, 2, 3 only.

Here again we find on examination, as is to be expected, that (1·8a) implies flat space-time and further that if (1·8b) is satisfied, we have:

$$R_{2332} = R_{3113} = R_{1221} = R_{1414} = R_{2424} = R_{3434} = (1/3)\varphi^2\lambda,$$

and all other curvature components vanish.

If we solve the differential equations obtained by taking

$$R_{\mu\nu} = 0, \qquad (\mu \neq \nu)$$

we get

$$\varphi^{-\frac{1}{2}} = f_1(x) + f_2(y) + f_3(z) + f_4(t) \tag{5·3}$$

where f_1, f_2, f_3 and f_4 are arbitrary functions. The equations (1·8b) mean four additional equations whose solution yields finally:

$$\varphi^{-\frac{1}{2}} = A(x^2+y^2+z^2-t^2) + B_1x + B_2y + B_3z + B_4t + C, \tag{5·4}$$

where

$$\lambda/3 = 4AC - (B_1^2 + B_2^2 + B_3^2) + B_4^2,$$

A, B_1, B_2, B_3, B_4 and C being arbitrary constants. Equations (1·8c) give the following values of φ, p and ϱ in the absence of mass-motion:

$$\varphi^{-\frac{1}{2}} = A(x^2+y^2+z^2) + B_1x + B_2y + B_3z + C + F(t), \tag{5·5}$$

$$\lambda - \kappa p = 3(\dot{F}^2 + 4AF - \alpha) - 2\varphi^{-\frac{1}{2}}(2A + F), \tag{5·7}$$

$$\lambda + \kappa\varrho = 3(F^2 + 4AF - \alpha) \tag{5·7}$$

where

$$\alpha = -4AC + B_1^2 + B_2^2 + B_3^2,$$

A, B_1, B_2, B_3 and C being arbitrary constants and F an arbitrary function of t.

6. General Observations. Since Liapounoff established the result that an isolated homogeneous fluid mass assumes a spherical form under the Newtonian law of gravitation, astronomical research has brought to our notice that large systems of matter have irregular forms, spheroidal, flat, etc., either because a sufficient time since their creation has not elapsed for them to assume the spherical form or because relativistic forces and other considerations including thermodynamical conditions and electromagnetic forces come into play. Various complicated gravitational situations are presented by star clusters and the nebulae. These can be represented by an internal line-element and an external line-element. The question of the boundary conditions to be satisfied has not yet been satisfactorily settled. It is accepted that the metric potentials must be continuous at the boundary. But this condition is not sufficient. It is worthwhile, therefore, to undertake a programme of investigation, in the first instance, concerning possible gravitational metrics having a certain form Some of the solutions thus obtained could provide either internal metrics or external metrics. We have reported here the first instalment of our results obtained in this connection. We are also looking into the question of the boundary conditions.

Various mathematical questions arose in the course of our investigation. One may be mentioned. We have remarked in the introduction that for the line-element (1·1), the result (1·6) holds good. We took up the reverse question. Does there exist non-trivial gravitational field that is one for which

$$R_{\mu\nu} = 0,$$

so that only $R_{\lambda\mu\nu\sigma} \neq 0, \quad (\lambda, \mu, \nu, \sigma \neq),$

the rest of the twenty components being zero? A simple calculation shows that since

$$R_{\mu\nu} \equiv g^{\lambda\sigma} R_{\lambda\mu\nu\sigma} = 0,$$

we must have $R_{\lambda\mu\nu\sigma} = 0, \quad (\lambda, \mu, \nu, \sigma \neq).$

It has to be admitted, of course, that the last condition is not a tensor condition.

We are grateful to Professor V V. Narlikar for having suggested the problem and guided us in the course of the investigation. Many of the results stated here are verifications of his conjectures.

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ON THE ORDER OF THE CESÀRO MEANS OF A SERIES CONJUGATE TO FOURIER SERIES

By

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1.1. Let

$$\sum_{n=1}^{\infty}(b_n \cos n\theta - a_n \sin n\theta) = \Sigma B_n(\theta) \tag{1.1.1}$$

be the conjugate series of a Fourier series associated with $f(\theta)$ which is integrable in the sense of Lebesgue over $(-\pi, \pi)$ and periodic outside this range with period 2π. The r^{th} derived series of the conjugate series (1.1.1) is

$$\sum_{n=1}^{\infty}\frac{d^r}{d\theta^r}(b_n \cos n\theta - a_n \sin n\theta). \tag{1.1.2}$$

We write

$$\psi(t) = \tfrac{1}{2}\{f(x+t)-f(x-t)\};$$

$$\theta(t) = \psi(t) - l;$$

$$P(t) = \sum_{p=0}^{r-1}\theta_p \frac{t^p}{p!},$$

θ_p's being constants independent of t;

$$h(t) = \frac{1}{2t^r}\left[\{f(x+t)-P(t)\}-(-1)^r\{f(x-t)-P(-t)\}\right]\dagger;$$

$$\Psi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\psi(u)\,du, \qquad (\alpha > 0),$$

$$\Psi_0(t) = \psi(t);$$

$$\psi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Psi_\alpha(t), \qquad (\alpha \geq 0);$$

and define $\Theta_\alpha(t)$, $\theta_\alpha(t)$, $H_\alpha(t)$, $h_\alpha(t)$ in a similar way.

Also we write

$$\overline{V}_\alpha(x) = \int_0^1 (1-t)^{\alpha-1}\sin xt\,dt, \qquad (\alpha \geq 0)$$

$$\overline{J}{}^{\alpha}_{\beta}(x) = \frac{(-1)^{h+1}x^{h+1}}{\Gamma(h+1-\alpha)\Gamma(\alpha+1)}\int_1^{\infty}(t-1)^{h-\alpha}\,\overline{V}{}^{h+1}_{1+\beta}(xt)\,dt, \qquad (\beta \geq \alpha \geq 0);$$

where h is the greatest integer not greater than α.

[†] $h(t)$ is supposed to be integrable (L).

If $\theta_\alpha(t)$, defined as above, tends to zero as $t \to 0$ then we say that $f(\theta)$ has the generalized jump l in mean of order α, at $\theta = x$.

1.2. Definition. [Riesz (1909); Chandrasekharan and Minakshisundaram (1952)] Let $\sum_1^\infty u_n$ be a given infinite series and λ_n a positive steadily increasing function of n, tending to ∞ with n. We write

$$A_\lambda(\omega) = A_\lambda^{0}(\omega) = \sum_{\lambda_n < \omega} u_n$$

$$A_\lambda^{r}(\omega) = \sum_{\lambda_n < \omega} (\omega - \lambda_n)^r u_n$$

and

$$C_\lambda^{r}(\omega) = A_\lambda^{r}(\omega)/\omega^r.$$

The series $\sum_1^\infty u_n$ is said to be summable (R, λ, r), $r \geq 0$ to sum s if $C_\lambda^r(\omega)$ exists and tends to s as $\omega \to \infty$.

1.3. In the year 1920 Lukacs [Lukacs, 1920] proved that if $\theta(t) \to 0$ as $t \to 0$, then

$$\bar{S}_n(x) \sim \frac{2l}{\pi} \log n,$$

$\bar{S}_n(x)$ being the n^{th} partial sum of the conjugate series (1.1.1) at $\theta = x$. In particular $\bar{S}.\,(x) = o[\log n]$ for almost all values of x. The object of the present paper is to extend the above result so as to be applicable to the order of the partial sums of the Cesàro means of the conjugate series (1.1.1) at $\theta = x$ and to frame examples showing that the result obtained is the *best possible**. Our theorem, given in §2, includes even a rather generalized form of the result of Lukacs as a particular case, We also obtain an analogous result for the order of the Cesaro means of the r^{th} derived series (1.1.2) at $\theta = x$ in §4.

From the theorems of this paper, in conjunction with certain known results, we have deduced like corollaries a number of results in §5, giving sufficient conditions for the summability (R, λ, k) of the conjugate series and its successively derived series for different values of λ and k.

2.1. We establish the following theorem.

Theorem 1. *If $\bar{S}_\alpha(\omega)$ denotes the (C, α) mean of (1.1.1) at $\theta = x$, and if*

$$\int_t^\pi \frac{|\theta_\alpha(t)|}{t}\, dt = o\left[\left(\log \frac{1}{t}\right)^{p+1}\right], \quad (\alpha \geq 0,\ -1 < p < \infty),$$

as $t \to 0$, then we have

$$\bar{S}_\alpha(\omega) \sim \frac{2l}{\pi} \log \omega \qquad (-1 < p \leq 0)$$

$$= o[(\log \omega)^{p+1}], \qquad (p > 0).$$

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* The analogous results for the case of Fourier series have been already obtained by the authoress of this paper and they are under publication. The paper containing these will be referred to as (A).

2.2. We make use of the following lemma due to Bosanquet (1934).

Lemma 1. *If* $\bar{S}_\beta(\omega)$ *denotes the* (C, β) *mean of the conjugate series* (1.1.1) *at* $\theta = x$, *then*

$$\bar{S}_\beta(\omega) = \frac{2}{\pi}\,\omega \int_0^\infty \psi_\alpha(t)\bar{J}^\alpha_\beta(\omega t)\,dt, \qquad (\beta \geq \alpha \geq 0),$$

where $\bar{J}^\alpha_\beta(x)$ *and its derivatives are bounded for* $x \geq 0$, *and for large values of* x

$$\bar{J}^\alpha_\beta(x) = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)}\;\frac{\sin\{x - \frac{1}{2}\pi(\alpha+\beta+1)\}}{x^{1+\beta-\alpha}} + \frac{1}{x} + O\Big[\frac{1}{x^2}\Big].$$

Also if $\eta > 0$, *then*

$$\omega \int_\eta^\infty \psi_\alpha(t)\{\bar{J}^\alpha_\beta(\omega t) - \frac{1}{\omega t}\}\,dt = o(1).$$

2.3. Proof of Theorem 1. We have by Lemma 1, writing $\theta^*_\alpha(t) = \int_t^\pi \frac{|\theta_\alpha(u)|}{u}\,du$,

$$\begin{aligned}\bar{S}_\alpha(\omega) &= \frac{2}{\pi}\,\omega \int_0^\infty \psi_\alpha(t)\bar{J}^\alpha_\alpha(\omega t)\,dt \\ &= \frac{2}{\pi}\,\omega \int_0^\pi \theta_\alpha(t)\bar{J}^\alpha_\alpha(\omega t)\,dt + \frac{2l}{\pi}\,\omega \int_0^\pi \bar{J}^\alpha_\alpha(\omega t)\,dt + \frac{2}{\pi}\int_\pi^\infty \frac{\psi_\alpha(t)}{t}\,dt + o(1).\end{aligned} \tag{2.3.1}$$

Now by Lemma 1

$$\begin{aligned}&\omega \int_0^\pi \theta_\alpha(t)\bar{J}^\alpha_\alpha(\omega t)\,dt \\ &= O\Big[\omega \int_0^{\pi\alpha/\omega} t\frac{|\theta_\alpha(t)|}{t}\,dt\Big] + \int_{\pi\alpha/\omega}^\pi \frac{\theta_\alpha(t)}{t}\{1 - \cos(\omega t - \pi\alpha)\}\,dt + O\Big[\int_{\pi\alpha/\omega}^\pi \frac{|\theta_\alpha(t)|}{\omega t^2}\,dt\Big] \\ &= O\Big[\omega\{t\theta_\alpha^*(t)\}_{t=0}^{\pi\alpha/\omega}\Big] + O\Big[\omega\int_0^{\pi\alpha/\omega} \theta_\alpha^*(t)\,dt\Big] + O\Big[\int_{\pi\alpha/\omega}^\pi \frac{|\theta_\alpha(t)|}{t}\,dt\Big] + o[(\log\omega)^{p+1}] \\ &= o[(\log\omega)^{p+1}].\end{aligned} \tag{2.3.2}$$

Next, by second mean value theorem,

$$\begin{aligned}\frac{2l}{\pi}\,\omega\int_0^\pi \bar{J}^\alpha_\alpha(\omega t)\,dt &= O\Big[\omega\int_0^{\pi\alpha/\omega} dt\Big] + \frac{2l}{\pi}\int_{\pi\alpha/\omega}^\pi \frac{1}{t}\,dt - \frac{2l}{\pi}\int_{\pi\alpha/\omega}^\pi \frac{\cos(\omega t - \pi\alpha)}{t}\,dt + O\Big[\int_{\pi\alpha/\omega}^\pi \frac{1}{\omega t^2}\,dt\Big] \\ &= O(1) + \frac{2l}{\pi}\log\omega.\end{aligned} \tag{2.3.3}$$

Also it is easily seen that

$$\int_{\pi}^{\infty} \frac{\psi_{\alpha}(t)}{t}\, dt = O(1). \tag{2.3.4}$$

Hence we have from (2.3.1)……(2.3.4)

$$\bar{S}_{\alpha}(\omega) = \frac{2l}{\pi} \log \omega + o[(\log \omega)^{p+1}],$$

which completes the proof.

2.4. Making use of Theorem 1 and the following well known identity [Hardy (1910); Kogbetliantz (1931)]

$$\tau_{\alpha}(\omega) = \alpha\{S_{\alpha-1}(\omega) - S_{\alpha}(\omega)\}, \qquad (\alpha > 0),$$

where $\tau_{\alpha}(\omega)$ and $S_{\alpha}(\omega)$ denote the (C, α) mean of the sequence $\{nu_n\}$ and the series Σu_n, respectively, we obtain the following theorem.

Theorem 2. *If*

$$\int_{t}^{\pi} \frac{|\theta_{\alpha}(t)|}{t}\, dt = o\left[\left(\log \frac{1}{t}\right)^{p+1}\right], \qquad (-1 < p < \infty, \alpha \geq 0),$$

as $t \to 0$, *then we have*

$$\tau_{\alpha+1}(\omega) = o[(\log \omega)^{p+1}],$$

where $\tau_{\alpha+1}(\omega)$ *denotes the* $(C, \alpha+1)$ *mean of the sequence* $\{nB_n(x)\}$.

3.1. We shall now show that the results of Theorems 1 and 2 are the *best possible.* We consider the case $l = 0$ of Theorem 1 only, the analysis for the general case and Theorem 2 being the same. Firstly we shall show that:

There is an odd periodic integrable function $\psi(t)$ *such that*

$$\psi_{\alpha}(t) = o\left[\left(\log \frac{1}{t}\right)^{p}\right], \qquad (\alpha \geq 0, -1 < p < \infty),$$

as $t \to 0$, *and at* $t = 0$

$$\bar{S}_{\alpha}(\omega) > \xi_{\omega}(\log \omega)^{p+1} \tag{3.1.1}$$

for any ξ_{ω} *tending to zero as* $\omega \to \infty$, *however slowly, and for arbitrary large values of* ω.

Secondly we show that:

There is an odd periodic integrable function $\psi(t)$ *such that*

$$\psi_{\alpha}(t) = O\left[\left(\log \frac{1}{t}\right)^{p}\right], \qquad (\alpha \geq 0, - < p < \infty),$$

and

$$\int_{0}^{t} \psi_{\alpha}(u)\, du = o\left[t\left(\log \frac{1}{t}\right)^{p}\right],$$

as $t\to 0$, *but at* $= 0$ $\quad \bar S_\alpha(\omega) \neq o[(\log\omega)^{p+1}] \quad$ (3.1.2)

for arbitrary large values of ω.

3.2. It is easy to see that, if

$$\psi_\alpha(t) = o\left[\left(\log\frac{1}{t}\right)^p\right], \qquad (-1 < p < \infty),$$

as $t\to 0$, then we have

$$\int_0^t |\psi_\alpha(t)|\,dt = o\left[t\left(\log\frac{1}{t}\right)^p\right],$$

as $t\to 0$, but not vice versa, and if

$$\int_0^t |\psi_\alpha(t)|\,dt = o\left[t\left(\log\frac{1}{t}\right)^p\right], \qquad (-1 < p < \infty),$$

as $t\to 0$, then

$$\int_t^\pi \frac{|\psi_\alpha(t)|}{t}\,dt = o\left[\left(\log\frac{1}{t}\right)^{p+1}\right],$$

as $t\to 0$, but not vice versa.

Thus the statements (3.1.1), (3.1.2) show that in Theorem 1, neither order estimations of $\bar S_\alpha(\omega)$ can be modified, nor the order conditions of $\psi_\alpha(t)$ can be replaced by less restrictive conditions of a certain type.

3.3. We shall require the following Lemma.

Lemma 2. *If*

$$\int_0^t |\psi_\alpha(u)|\,du = o\left[t\left(\log\frac{1}{t}\right)^p\right], \qquad (\alpha \geq 0,\ -1 < p < \infty)$$

as $t\to 0$, *then we have*

$$\bar S_\alpha(\omega) = o\,[(\log\omega)^p] + \frac{2}{\pi}\int_\pi^\infty \frac{\psi_\alpha(t)}{t}\,dt - \frac{2}{\pi}\int_{\pi\alpha/\omega}^\pi \psi_\alpha(t)\frac{\cos(\omega t - \pi_\alpha)}{t}\,dt + o(1).$$

Proof. We have by Lemma 1

$$\begin{aligned}\bar S_\alpha(\omega) &= \frac{2}{\pi}\,\omega\int_0^\infty \psi_\alpha(t)\bar J^\alpha_\alpha(\omega t)\,dt\\ &= O\left[\omega\int_0^{\pi\alpha/\omega} |\psi_\alpha(t)|\,dt\right] + \frac{2}{\pi}\int_{\pi\alpha/\omega}^\pi \psi_\alpha(t)\,\frac{1-\cos(\omega t - \pi\alpha)}{t}\,dt + \frac{2}{\pi}\int_\pi^\infty \frac{\psi_\alpha(t)}{t}\,dt + o(1)\\ &= o[(\log\omega)^p] + \frac{2}{\pi}\int_{\pi\alpha/\omega}^\infty \frac{\psi_\alpha(t)}{t}\,dt - \frac{2}{\pi}\int_{\pi\alpha/\omega}^\pi \psi_\alpha(t)\,\frac{\cos(\omega t - \pi\alpha)}{t}(dt + o(1),\end{aligned}$$

which is the required

3.4. To demonstrate (3.1.1) we construct the following example:

Example 1. Consider first α *to be a non-integral number. Let* $[\alpha]$ *and* $\{\alpha\}$ *denote its integral and fractional parts, respectively. Let* $h = [\alpha]+1$. *Take the sequence of integers* $R_0, R_1, R_2, \ldots\ldots\ldots$ *monotonically tending to infinity such that*

$$R_0 = A$$

where A *is a sufficiently large integer, and*

$$R_i = \{[(R_{i-1})^{(\{\alpha\}-\epsilon)/(1+\alpha)}]/\alpha+1\}R_{i-1},$$

where the inner brackets denote the integral part of p_i [*]. *Let*

$$\alpha_i = \pi\alpha/R_i,$$

and let $\{c_i\}$ *be a sequence tending to zero as* $i\to\infty$, *such that* $\xi_i/c_i\to 0$ *as* $i\to\infty$, *for any given sequence* $\{\xi_i\}$ *tending to zero as* $i\to\infty$, *however slowly. We denote the interval* $[\alpha_i, \alpha_{i-1}]$ by I_i, *and define an odd periodic function* $\sigma(t)$ *such that*

$$\Gamma(\alpha+1)\sigma(t) = c_{n_i}t^{\alpha}\left(\log\frac{1}{t}\right)^{p}\cos(R_{n_i}t-\pi\alpha),$$

for t *lying in* I_{n_i}, *and*

$$\sigma(t) = 0,$$

for t *lying every where else in* $(0, \pi)$, *the sequence* $\{n_i\}$ *consisting of positive integers monotonically tending to infinity, taken with sufficient large gaps.*

We define

$$\psi(t) = \frac{1}{\Gamma(1-\{\alpha\})}\int_0^t (t-u)^{-\{\alpha\}}\sigma^h(u)du.$$

In order to show that $\psi(t)$ is integrable (L) it will be sufficient to show that $\sigma^h(t)$ is integrable (L). Differentiating h times we get

$$\frac{\sigma^h(t)}{c_{n_i}} = O[t^{\alpha-\epsilon_1}(R_{n_i})^h] + O[(R_{n_i})^{h-1}t^{\alpha-1-\epsilon_1}] + \ldots\ldots + O[(R_{n_i})^{h-m}t^{\alpha-m-\epsilon_1}] + \ldots\ldots$$
$$+ O[t^{\alpha-h-\epsilon_1}],\ (0<\epsilon_1<1),$$

or

$$\sigma^h(t) = P_1+P_2+\ldots\ldots+P_{m+1}+\ldots\ldots+P_{h+1},$$

say. It is easily verified [†] that for $1\leq m\leq h$,

$$\int_0^{\pi} P_{m+1}dt = \sum_{i=0}^{\infty}\int_{I_{n_i}} P_{m+1}dt = O(1),$$

which shows that $\sigma^h(t)$ and so consequently $\psi(t)$ is integrable (L). Again by a theorem of fractional integrals we have

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[*] $p_i = (R_{i-1})^{(\{\alpha\}-\epsilon)/(1+\alpha)}$.

[†] The analysis is exactly similar to that given in (A). In the remaining paper the unnecessary repetition of arguments is avoided as the details of analysis run parallel to that in case (A).

$$\psi_\alpha(t) = \Gamma(\alpha+1)t^{-\alpha}\Psi_\alpha(t) = \frac{\alpha}{t^\alpha}\int_0^t (t-u)^{\alpha-1}\psi(u)\,du$$

$$= c_{n_i}\left(\log\frac{1}{t}\right)^p \cos(R_{n_i}t-\pi\alpha)$$

for values of t lying in I_{n_i}, and

$$\psi_\alpha(t) = 0,$$

everywhere else in $(0, \pi)$. It is easily seen that

$$\psi_\alpha(t) = o\left[\left(\log\frac{1}{t}\right)^p\right].$$

We now show that

$$\bar{S}_\alpha(\omega) > \xi_\omega(\log\omega)^{p+1}.$$

According to Lemma 2 we have

$$\bar{S}_\alpha(\omega) = o\,[(\log\omega)^{p+1}] - \frac{2}{\pi}\int_{\pi\alpha/\omega}^{\pi}\psi_\alpha(t)\frac{\cos(\omega t-\pi\alpha)}{t}dt,$$

or

$$\bar{S}_\alpha(R_{n_i}) = o\,[(\log R_{n_i})^{p+1}] - J, \tag{3.4.1}$$

say. We have

$$J = \frac{2}{\pi}\int_{\pi\alpha/R_{n_i}}^{\pi\alpha/R_{n_i}-1} C_{n_i}\frac{\cos^2(R_{n_i}t-\pi\alpha)}{t}\left(\log\frac{1}{t}\right)^p dt + \frac{2}{\pi}\int_{\pi\alpha/(R_{n_i}-1)}^{\pi}\psi_\alpha(t)\frac{\cos(R_{n_i}t-\pi\alpha)}{t}dt$$

$$= J_1 + J_2, \tag{3.4.2}$$

say. Now

$$J_1 = \frac{C_{n_i}}{\pi}\int_{\pi\alpha/R_{n_i}}^{\pi\alpha/R_{n_i}-1}\frac{(\log 1/t)^p}{t}dt + \frac{C_{n_i}}{\pi}\int_{\pi\alpha/R_{n_i}}^{\pi\alpha/R_{n_i}-1}\frac{\cos(2R_{n_i}t-2\pi\alpha)}{t}\left(\log\frac{1}{t}\right)^p dt$$

$$> \xi_{n_i}(\log R_{n_i})^{p+1}. \tag{3.4.3}$$

Next

$$J_2 = \frac{2}{\pi}\int_{\pi\alpha/R_{n_i}-1}^{\pi}\psi_\alpha(t)\frac{\cos(R_{n_i}t-\pi\alpha)}{t}dt = \frac{2}{\pi}\int_{\pi\alpha/R_{n_{i-1}}}^{\pi}\psi_\alpha(t)\frac{\cos(R_{n_i}t-\pi\alpha)}{t}dt$$

as

$$\psi_\alpha(t) = 0 \text{ in } [\pi\alpha/R_{n_i}-1, \pi\alpha/R_{n_{i-1}}].$$

Or

$$J_2 = \sum_{s=1}^{i-1}\int_{\pi\alpha/R_{n_s}}^{\pi\alpha/R_{n_s}-1} C_{n_s}\frac{\cos\{(R_{n_i}+R_{n_s})t-2\pi\alpha\}+\cos(R_{n_i}-R_{n_s})t}{t}\left(\log\frac{1}{t}\right)^p dt$$

$$= O\left[\sum_{s=1}^{i-1}\frac{R_{n_s}(\log R_{n_s})^p}{R_{n_i}-R_{n_s}}\right]$$

$$= o(1). \tag{3.4.4}$$

Collecting all our results (3.4.1)......(3.4.4) we get

$$\bar{S}_\alpha(\omega) > \xi_\omega (\log \omega)^{p+1},$$

which establishes the statement (3.1.1) for non-integral values of σ. For the case when α is an integer the above analysis becomes simpler by putting $\alpha = h$, and defining

$$R_i = \{[(R_{i-1})^{(1-\epsilon)/(1+\epsilon)}]/\alpha + 1\}R_{i-1}, \qquad (0 \overline{<} \epsilon < 1),$$

Example 2. First consider α *to be a non-integral number and let* $[\alpha]$ *and* $\{\alpha\}$ *denote its integral and fractional parts respectively, and let* $h = [\alpha] + 1$. *Let* $\{x_i\}$ *be a sequence of positive increasing integers tending to infinity sufficiently slowly. Also take the sequence of integers* $R_0, R_1, R_2 \ldots\ldots$ *monotonically tending to infinity such that*

$$R_0 = K,$$

$$R_i = \{[[(R_{i-1})^{(\{\alpha\}-\delta)/h}]]/\alpha + 1\}R_{i-1}, \qquad (0 < \delta < \{\alpha\}),$$

where by $[[p_i]]$ *we denote the nearest integer greater than* p_i, *such that* $[[p_i]]/\alpha$ *is also an integer, where* $p_i = (R_{i-1})^{(\{\alpha\}-\delta)/h}$.

We write

$$\mu_i = \frac{x_i}{\alpha} + 1,$$

$$M_i = \mu_i R_i,$$

$$\alpha_i = \frac{\pi\alpha}{R_i},$$

$$C_i = \left(\log \frac{R_{i-1}}{\pi\alpha}\right)^p,$$

and denote the interval $[\alpha_i, \alpha_{i-1}]$ *by* I_i. Let $\sigma(t)$ *be an odd periodic function such that*

$$\Gamma(\alpha+1)\sigma(t) = C_{n_i} t^\alpha \cos(M_{n_i} t - \pi\alpha),$$

for values of t *lying in* I_{n_i} *where* n_i *is a sequence of positive increasing integers taken with sufficient large gaps, and*

$$\sigma(t) = 0,$$

everywhere else in $(0, \pi)$.

We define $$\psi(t) = \frac{1}{\Gamma(1-\{\alpha\})} \int_0^t (t-u)^{-\{\alpha\}} \sigma^h(u)\,du.$$

It is easy to see † as in Example 1 that $\sigma^h(t)$ and so consequently $\psi(t)$ is integrable (L). We have as before

$$\psi_\alpha(t) = C_{n_i} \cos(M_{n_i} t - \pi\alpha),$$

---

† See (A).

for values of t lying in I_{n_i}, and

$$\psi_\alpha(t) = 0,$$

everywhere else in $(0, \pi)$.

We have if i belongs to the sequence $\{n_p\}$,

$$\int_{\pi\alpha/R_i}^{\pi\alpha/R_{i-1}} \psi_\alpha(t)dt = \frac{C_i}{M_i}\left[\sin\left(\frac{M_i\pi\alpha}{R_{i-1}} - \pi\alpha\right) - \sin\left(\frac{M_i}{R_i}\pi\alpha - \pi\alpha\right)\right] = 0.$$

Hence if t lies in I_{n_i}, then

$$\int_0^t \psi_\alpha(u)du = \int_{\pi\alpha/Rn_i}^t C_{n_i}\cos(M_{n_i}t - \pi\alpha)dt + 0$$

$$= \frac{C_{n_i}}{M_{n_i}}\sin(M_{n_i}t - \pi\alpha)$$

$$= o\left[t\left(\log\frac{1}{t}\right)^p\right].$$

Again

$$\psi_\alpha(t) = O(C_{n_i})$$

$$= O\left[\left(\log\frac{1}{t}\right)^p\right].$$

We now show that

$$\overline{S}_\alpha(\omega) \neq o[(\log\omega)^{p+1}].$$

Proceeding as before we have

$$\overline{S}_\alpha(M_{n_i}) = o[(\log M_{n_i})^{p+1}] - J, \tag{3.5.1}$$

where

$$J = \frac{2}{\pi}\int_{\pi\alpha/Mn_i}^{\pi} \psi_\alpha(t)\frac{\cos(M_{n_i}t - \pi\alpha)}{t}dt$$

$$= \frac{2}{\pi}\int_{\pi\alpha/Mn_i}^{\pi\alpha/Rn_i} + \frac{2}{\pi}\int_{\pi\alpha/Rn_i}^{\pi\alpha/R_{n_i-1}} + \frac{2}{\pi}\int_{\pi\alpha/R_{n_i-1}}^{\pi}$$

$$= J_1 + J_2 + J_3, \tag{3.5.2}$$

say. Now as we take the sequence of integers $\{n_i\}$ with sufficient large gaps we can take n_i such that

$$(\pi\alpha/R_{n_{i+1}-1}) < (\pi\alpha/M_{n_i}),$$

so that $(\pi\alpha/M_{n_i})$ lies neither in the interval $I_{n_{i+1}}$ nor in the interval I_{n_i}. Thus we get $\psi_\alpha(t) = 0$ in the interval $[\pi\alpha/M_{n_i}, \pi\alpha/R_{n_i}]$, and therefore

$$J_1 = 0. \tag{3.5.3}$$

Next

$$J_2 = \frac{C_{n_i}}{\pi} \int_{\pi\alpha/R_{n_i}}^{\pi\alpha/R_{n_i-1}} \frac{1}{t}\, dt + \frac{C_{n_i}}{\pi} \int_{\pi\alpha/R_{n_i}}^{\pi\alpha/R_{n_i-1}} \frac{\cos\{2(M_{n_i}t-\pi\alpha)t\}}{t}\, dt$$

$$= \frac{C_{n_i}}{\pi} \log (R_{n_i}/R_{n_i-1}) + O[C_{n_i}R_{n_i}/M_{n_i}]$$

$$= A(\log M_{n_i})^{p+1}[1+o(1)],$$

where A is a constant >0. This shows that

$$J_2 \neq o[(\log M_{n_i})^{p+1}]. \tag{3.5.4}$$

Again proceeding as in the previous example, we have

$$J_3 = o[(\log M_{n_i})^{p+1}]. \tag{3.5.5}$$

Thus we have from (3.5.1)......(3.5.5), putting $\omega = M_{n_i}$,

$$\bar{S}_\alpha(\omega) \neq o[(\log \omega)^{p+1}]$$

which demonstrates the statement (3.1 2) for non-integral values of α. For the case when α is an integer, the above analysis becomes simpler by putting $\alpha = h$ and

$$R_i = \{[[(R_{i-1})^{(1-\delta)/\alpha}]]/\alpha+1\}R_{i-1}, \qquad (0<\delta<1).$$

Thus the statement (3.1.2) is completely demonstrated.

4. Theorem 3. *If*

$$\int_t^\pi \frac{|h_\delta(t)|}{t} dt = o\left[\left(\log \frac{1}{t}\right)^{p+1}\right] \qquad (\delta \geq 0,\ -1<p<\infty),$$

as $t \to 0$ *then we have*

$$\bar{\mathcal{B}}_r^{r+\delta}(\omega) = o[(\log \omega)^{p+1}],$$

as $\omega \to \infty$, $\bar{\mathcal{B}}_r^{r+\delta}(\omega)$ *being the* $(C,\ r+\delta)$ *mean of the series* (1.1 2) *at* $\theta = x$.

The above result is obtained by virtue of the fact that

$$A_r^{\delta+i}(\omega) = o[(\log \omega)^{p+1}], \qquad (i = 0, 1, 2, \ldots .r),$$

by theorem 1, where $A_r^{\delta+i}(\omega)$ denotes the $(i+\delta)^{\text{th}}$ Cesàro mean of the conjugate series corresponding to $h(t)$ at $t=0$, and the following Lemma due to Misra (1946).

Lemma 3. *If* $\bar{\mathcal{B}}_r^k(\omega)$ *denotes the* (C, k) *mean, for* $k \geq r$, *of the* r^{th} *derived series of the allied series* (1.1.1) *at* $\theta = x$, *and* $A_r^{k-i}(\omega)$ *denotes the* $(k-i)^{\text{th}}$ *Cesàro mean of the conjugate series corresponding to* $h(t)$ *at* $t = 0$, *then we have*

$$\bar{\mathcal{B}}_r^k(\omega) - r!s = (-1)^r \sum_{i=0}^{r} (-1)^i C_i^r \frac{\Gamma(1+k+i)}{\Gamma(1+k+i-r)} \{\ {}_r^{k+i-r}(\omega) - s\},$$

where $$C_i^r = \frac{(r-i+1)(r-i+2)\dots(r)}{\Gamma(i+1)}.$$

The above Theorem 3 can also be put in the following form, which shows that a recently published result of Mohanty and Nanda (1955) [Proc. Amer. Math. Soc., 6, 594] is included in it:

Theorem 3′. *If*

$$\int_t^\pi \frac{|h_\delta(t)-l|}{t}dt = o\,[(\log\omega)^{p+1}], \qquad (\delta \geq 0,\ -1 < p \lesssim \infty),$$

as $t \to 0$, *then we have*

$$\overline{\mathcal{B}}_r^{r+\delta}(\omega) = \frac{2l}{\pi} r!\ \log\omega + o\{(\log\omega)^{p+1}\},$$

as $w \to \infty$. *

5.1. We give some corollaries in this section which are obtained directly from Theorems 1 and 3 and the following Lemmas.

Lemma 4 §. *(a) If* (C, α) *means of the series* $\sum_{r=1}^{\infty} u_r$ *are* $o[\log n]$ *as* $n \to \infty$, *for* $\alpha \geq 0$, *and if its* $(C, \alpha+1)$ *mean exists, then the series* $\sum_{r=1}^{\infty} \{u_r/\log(r+1)\}$ *is summable* (C, α).

(b) If the partial sums of the series $\sum_{r=1}^{\infty} u_r$ *are* $o[(\log n)^{p+1}]$ *and its* $(C, 1)$ *means are* $s + o[(\log n)^p]$, *as* $n \to \infty$, *for* $-1 \lesssim p < \infty$, *then the series* $\sum_{r=1}^{\infty} [u_r/\{\log(r+1)\}^{p+1}]$ *is convergent.*

Lemma 5. [Zygmund, 1925]. *If* $\Sigma a_n = o(R, n, \beta)$ *for some* β *and* $S_\alpha(\omega) = o[p^\alpha(\omega)]$ *where* $p(\omega)$ *is an increasing function which tends to* ∞ *as* $\omega \to \infty$ *and* $S_\alpha(\omega)$ *denotes the* (C, α) *mean of the series* Σa_n, *then*

$$\Sigma a_n = o(R, \mu, \alpha),$$

where $$\mu(x) = \exp\left[\int^{x} \frac{dt}{tp(t)}\right].$$

5.2. **Corollary 1.** *If*

$$\int_0^t |\psi_\alpha^1(t)|\,dt = o(t),$$

---

* This paragraph has been added in proof.

§ See (A)

and
$$\lim_{t\to 0}\frac{2}{\pi}\int_t^\infty \frac{\psi(u)}{u}du \to S(c),$$

then the series
$$\sum_{n=1}^{\infty}[(b_n\cos nx - a_n\sin nx)/\{\log(n+1)\}]$$

is summable (C, α), *and if*

$$\int_0^t |\psi(t)|\,dt = o\left[t\left(\log\frac{1}{t}\right)^p\right], \qquad (-1<p<\infty),$$

and
$$\lim_{t\to 0}\frac{2}{\pi}\int_t^\infty \frac{\psi(u)}{u}du \to S(c),$$

then the series

$$\sum_{n=1}^{\infty}[b_n\cos nx - a_n\sin nx)/\{\log(n+1)\}^{p+1}]$$

is convergent.

The result follows from Theorem 1 and Lemma 4.

Corollary 2. *If*

$$\int_0^t |h_\delta(t)|\,dt = o(t), \qquad (\delta \geq 0),$$

as $t\to 0$, *and*

$$\lim_{t\to 0}\frac{2}{\pi}\int_t^\infty \frac{h_\delta(t)}{t}dt \to S(c)$$

then the series
$$\sum_{n=1}^{\infty}\left[\left\{\frac{d^r}{d\theta^r}(b_n\cos n\theta - a_n\sin n\theta)\right\}/\{\log(n+1)\}\right]$$

is summable $(\mathbf{C}, r+\delta)$, *at* $\theta = x$.

The above result is easily obtained from Theorem 3 and Lemma 4.

Corollary 3. (*a*) *If*

$$\int_t^\pi \frac{|\psi_\alpha(t)|}{t}dt = o\left[\left(\log\frac{1}{t}\right)^\alpha\right], \qquad (\alpha > 0),$$

and
$$\lim_{t\to 0}\frac{2}{\pi}\int_t^\infty \frac{\psi(t)}{t}dt \to S(c),$$

then the conjugate series (1.1.1) *is summable* $(R, \log n, \alpha)$, *at* $\theta = x$ *to sum* S.

(*b*) *If*

$$\int_t^\pi \frac{|\psi_\alpha(t)|}{t}dt = o\left[\left(\log \frac{1}{t}\right)\right], \qquad (\alpha > 1),$$

as $t \to 0$, *and*

$$\operatorname*{Lim}_{t\to 0} \frac{2}{\pi}\int_t^\infty \frac{\psi(t)}{t}\,dt \to S(c),$$

then the conjugate series (1.1.1) *is summable* (R, μ, α) *at* $\theta = x$ *to sum* S, *where*

$$u(n) = \exp\{(\log n)^{1-1/\alpha}\}.$$

The result follows from Theorem 1 and Lemma 5.

Corollary 4. (*a*) *If*

$$\int_t^\pi \frac{|h_\delta(t)|}{t}dt = o\left[\left(\log \frac{1}{t}\right)^{r+\delta}\right], \qquad (\delta \geq 0),$$

and

$$\operatorname*{Lim}_{t\to 0} \frac{2}{\pi}\int_t^\infty \frac{h_\delta(t)}{t}dt \to S(c),$$

then the r^{th} *derived series of the conjugate series* (1.1.1) *is summable* $(R, \log n, r+\delta)$ *to sum* $r!\ S$ *at* $\theta = x$, *and*

(*b*) *If*

$$\int_t^\pi \frac{|h_\delta(t)|}{t}dt = o\left[\log \frac{1}{t}\right], \qquad (r+\delta \geq 1,\ \delta \geq 0),$$

as $t \to 0$, *and if*

$$\operatorname*{Lim}_{t\to 0} \frac{2}{\pi}\int_t^\infty \frac{h_\delta(t)}{t}dt \to S(c),$$

then the r^{th} *derived series of the conjugate series* (1.1.1) *is summable* $(R, \mu, r+\delta)$ *at* $\theta = x$, *to sum* $r!\ S$, *where*

$$\mu(n) = \exp\{(\log n)^{1-1/(r+\delta)}\}.$$

The result follows from Theorem 3 and Lemma 5.

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PROJECTIVE CURVATURE TENSOR AND RELATIVISTIC GRAVITATION

By

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Abstract. In this note the influence of the vanishing of the projective curvature tensor and of its divergence on a relativistic gravitational field is explored It is found that there is no gravitational field consistent with the first condition and that there is no perfect fluid distribution consistent with the second condition.

The present investigation arose out of a study of the projective curvature tensor W_{hijk} from the gravitational point of view. It has ten (Eisenhart, 1926, p. 135) algebraically independent components which are given by

$$W_{hijk} = R_{hijk} - \frac{1}{n-1}(g_{hk}R_{ij} - g_{hj}R_{ik}) \tag{1}$$

for a Riemannian V_n. It is well-known (Thomas, 1934, p. 217) that a necessary and sufficient condition for a Riemannian space to be a space of constant curvature is that it should be projectively flat, *i e.*,

$$W_{hijk} = 0$$

which leads to

$$R_{hijk} = \frac{1}{n-1}(g_{hk}R_{ij} - g_{hj}R_{ik}). \tag{2}$$

From (1) it is obvious that if R_{ij} and W_{hijk} vanish together, then $R_{hijk} = 0$. Hence if a part of a gravitational space which is free from matter has a constant curvature then the latter is zero.

Again if

$$R_{ij} \neq 0$$

but

$$R_{ij} - \tfrac{1}{2}g_{ij}R = -\kappa T_{ij} \tag{3}$$

and $W_{hijk} = 0$ we have from (2) for a Riemannian fourfold

$$R_{ij} = Rg_{ij}/4. \tag{4}$$

which is physically untenable. Hence in a Riemannian spacetime of constant curvature there can be no physically possible distributions of matter.

When we turn to the divergence of the projective curvature tensor we find that the conditions

$$W^{h}{}_{ijk,h} = 0 \tag{5}$$

are the same as

$$R^h{}_{ijk,h} = 0 \tag{6}$$

or as

$$R_{ij,k} = R_{ik,j}, \tag{7}$$

a comma (,) preceding a suffix denoting a covariant differentiation with respect to the corresponding variable. It is clear that these conditions always hold for a gravitational space free from matter. For a space-time occupied by matter we consider the following special case of a material distribution,

$$T_{ij} = (\varrho+p)v_i v_j - p g_{ij} \tag{8}$$

with

$$v_1 = v_2 = v_3 = 0,\ v_4 = 1,$$

v_μ being the flow vector. The equations (7) lead to

$$(T_{ij}-\tfrac{1}{2}Tg_{ij})_{,k} = (T_{ik}-\tfrac{1}{2}T_{ik})_{,j} \tag{10}$$

which stand for forty equations. For the above case of material distribution given by (8) four of the forty equations give us

$$\frac{\partial\varrho}{\partial r} = \frac{\partial p}{\partial r},\quad \frac{\partial\varrho}{\partial\theta} = \frac{\partial p}{\partial\theta},$$

$$\frac{\partial\varrho}{\partial\varphi} = \frac{\partial p}{\partial\varphi},\quad \frac{\partial\varrho}{\partial t} = \frac{\partial p}{\partial t}, \tag{11}$$

as a consequence of which we have

$$\varrho = p. \tag{12}$$

Equation (11) is physically untenable. Thus a perfect fluid distribution of the type (8) is incompatible with the condition $W^h{}_{ijk,k} = 0$ which conflicts with the physical limitations on ϱ and p appearing in T_{ij}.

The author's grateful thanks are due to Prof. V. V. Narlikar, Head of the Department of Mathematics, Banaras Hindu University, who suggested the above investigation and who had anticipated the results obtained herein.

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ON THE SLOW STEADY TWO-DIMENSIONAL FLOW OF A VISCOUS LIQUID IN A CHANNEL BOUNDED BY TWO NON-CONCENTRIC CIRCULAR ARCS

By

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Because of the non-linearity of the equations of motion of a viscous liquid very few exact solutions of these equtions have been obtained as yet. Jeffery (1915) has given the exact solutions of the two-dimensional flow of a viscous liquid in a circular canal with concentric circular boundaries, but no solutions of the problem is known when the boundary consists of two non-concentric circular arcs. In the present paper. the slow steady two-dimensional motion of a liquid between two non-concentric circles is determined.

1. Let us introduce a system of curvilinear orthogonal co-crdinates defined by

$$\alpha+i\beta = \log \frac{x+i(y+a)}{x+i(y-a)} \tag{1}$$

The cuvres α = constant are a set of co-axial circles having the points $(0, -a)$ and $(0, a)$ as limiting points, and the curves β = constant, the orthogonal set of circles passing through the limiting points. We have $\beta = \pi$ on the portion of the y-axis lying between the limiting points and $\beta = 0$ on the remainder of the y-axis.

From (1), we have

$$x+iy = ia \coth \tfrac{1}{2}(\alpha+i\beta)$$

Therefore

$$\frac{1}{h^2} = \left|\frac{d(x+iy)}{d(\alpha+i\beta)}\right|^2 = \left|\frac{a}{2\sinh^2\frac{1}{2}(\alpha+i\beta)}\right|^2$$

$$= \frac{a^2}{4\sinh^2\frac{1}{2}(\alpha+i\beta)\sinh^2\frac{1}{2}(\alpha-i\beta)}$$

$$= \frac{a^2}{(\cosh\alpha-\cos\beta)^2}$$

so that

$$h = \frac{\cosh\alpha-\cos\beta}{a} \tag{2}$$

2. When the motion is slow and steady, the two-dimensional equations of motion are

$$-\frac{\partial p}{\partial x}+\mu\nabla_1{}^2u = 0,\ -\frac{\partial p}{\partial y}+\mu\nabla_1{}^2v = 0 \tag{3}$$

the plane of the motion being taken as the (x, y) plane.

If ψ be the stream function of the liquid motion

$$u = -\frac{\partial\psi}{\partial y},\ \ v = \frac{\partial\psi}{\partial x} \tag{4}$$

Substituting in (3), we get

$$\frac{\partial p}{\partial x} = -\mu\frac{\partial}{\partial y}\nabla_1{}^2\psi,\ \ \frac{\partial p}{\partial y} = \mu\frac{\partial}{\partial x}\nabla_1{}^2\psi \tag{5}$$

These are Cauchy-Riemann differential equations, so that

$$\mu\nabla_1{}^2\psi+ip$$

is an analytic function of $x+iy$, and therefore of $\alpha+i\beta$.

Further, if we eliminate p between the equations (5), we see that ψ satisfies the equation

$$\nabla_1{}^4\psi = 0 \tag{6}$$

If we measure s along a fixed boundary, and n normal to it, we have on the assumption that there is no slipping on the boundary,

$$\frac{\partial\psi}{\partial n} = 0,\quad \frac{\partial\psi}{\partial s} = 0$$

on a fixed boundary. The second of these equations can be integrated with respect to s and gives $\psi =$ constant on a fixed boundary.

If $\alpha =$ constant be a fixed boundary, we have on using (2),

$$\frac{\partial\psi}{\partial n} = h\frac{\partial\psi}{\partial\alpha} = \frac{\partial}{\partial\alpha}(h\psi)-\frac{\psi}{a}\sinh\alpha$$

Therefore on a fixed boundary $\alpha =$ constant, we have

$$\left.\begin{aligned}&\psi = \text{constant},\\ \text{and}\quad &\frac{\partial}{\partial\alpha}(h\psi)-\frac{\psi}{a}\sinh\alpha = 0\end{aligned}\right\} \tag{7}$$

3. Let us assume as a solution of (6) (Jeffery, 1921)

$$h\psi = A_0\cosh\alpha+B_0\,\alpha\cosh\alpha+C_0\sinh\alpha+D_0\,\alpha\sinh\alpha+ [A_1\cosh 2\alpha+B_1+C_1\sinh 2\alpha+D_1\alpha]\cos\beta \tag{8}$$

If the motion takes place between two non-concentric circles $\alpha = \alpha_1$ and $\alpha = \alpha_2$, under pressure, the boundary conditions are

$$\psi = \psi_1 \quad \text{when} \quad \alpha = \alpha_1, \quad \text{and} \quad \psi = \psi_2 \quad \text{when} \quad \alpha = \alpha_2 \tag{9}$$

and

$$\frac{\partial}{\partial\alpha}(h\psi) - \frac{\psi}{a}\sinh\alpha = 0 \tag{10}$$

when

$$\alpha = \alpha_1 \text{ and when } \alpha = \alpha_2.$$

These conditions give

$$A_0 \cosh\alpha_1 + B_0\alpha_1 \cosh\alpha_1 + C_0 \sinh\alpha_1 + D_0\alpha_1 \sinh\alpha_1 = (\psi_1/a)\cosh\alpha_1$$

$$A_0 \cosh\alpha_2 + B_0\alpha_2 \cosh\alpha_2 + C_0 \sinh\alpha_2 + D_0\alpha_2 \sinh\alpha_2 = (\psi_2/a)\cosh\alpha_2$$

$$A_0 \sinh\alpha_1 + B_0(\cosh\alpha_1 + \alpha_1 \sinh\alpha_1) + C_0 \cosh\alpha_1 + D_0(\sinh\alpha_1 + \alpha_1 \cosh\alpha_1) = (\psi_1/a)\sinh\alpha_1$$

$$A_0 \sinh\alpha_2 + B_0(\cosh\alpha_2 + \alpha_2 \sinh\alpha_2) + C_0 \cosh\alpha_2 + D_0(\sinh\alpha_2 + \alpha_2 \cosh\alpha_2) = (\psi_2/a)\sinh\alpha_2$$

$$A_1 \cosh 2\alpha_1 + B_1 + C_1 \sinh 2\alpha_1 + D_1\alpha_1 = -\psi_1/a$$

$$A_1 \cosh 2\alpha_2 + B_1 + C_1 \sinh 2\alpha_2 + D_1\alpha_2 = -\psi_2/a$$

$$2A_1 \sinh 2\alpha_1 + 2C_1 \cosh 2\alpha_1 + D_1 = 0$$

$$2A_1 \sinh 2\alpha_2 + 2C_1 \cosh 2\alpha_2 + D_1 = 0$$

Solving these equations, we get

$$A_0 = \frac{\psi_1+\psi_2}{2a} - \tfrac{1}{2}K[\alpha_1^2 - \alpha_2^2 + \alpha_2 \sinh 2\alpha_1 - \alpha_1 \sinh 2\alpha_2 - \sinh(\alpha_1 - \alpha_2)\sinh(\alpha_1+\alpha_2)] \tag{11·1}$$

$$B_0 = K[\alpha_1 - \sigma_2 + \sinh(\alpha_1 - \alpha_2)\cosh(\alpha_1+\alpha_2)] \tag{11·2}$$

$$C_0 = K[\alpha_2 \cosh^2\alpha_1 - \alpha_1 \cosh^2\alpha_2 - \sinh(\alpha_1-\alpha_2)\cosh\alpha_1 \cosh\alpha_2] \tag{11·3}$$

$$D_0 = -K \sinh(\alpha_1 - \alpha_2)\sinh(\alpha_1+\alpha_2) \tag{11·4}$$

$$A_1 = K' \sinh(\alpha_1+\alpha_2) \tag{11·5}$$

$$B_1 = -\frac{\psi_1+\psi_2}{2a} - K'(\alpha_1+\alpha_2)\cosh(\alpha_1-\alpha_2) \tag{11·6}$$

$$C_1 = -K' \cosh(\alpha_1+\alpha_2) \tag{11.7}$$

$$D_1 = 2K' \cosh(\alpha_1-\alpha_2) \tag{11.8}$$

where

$$K = \frac{\psi_1 - \psi_2}{a[(\alpha_1-\alpha_2)^2 - \sinh^2(\alpha_1-\alpha_2)]} \tag{11·9}$$

$$K' = -\frac{\psi_1-\psi_2}{2a[(\alpha_1-\alpha_2)\cosh(\alpha_1-\alpha_2)-\sinh(\alpha_1-\alpha_2)]} \tag{11·10}$$

when the total flux across a section of the channel is prescribed, $\psi_1-\psi_2$ is known, so that the coefficients are all known except A_0 and B_1 which are indeterminate to the extent of the additive terms $\frac{1}{2}(\psi_1+\psi_2)/a$ and $-\frac{1}{2}(\psi_1+\psi_2)/a$.

But since we can add any constant term to ψ without altering the liquid motion, we adjust the constant in such a way that $\psi_1+\psi_2=0$. Then A_0 and B_1 become completely determinate.

4. To complete the solution of the problem it remains to determine the mean pressure p at any point. This is determined from the fact that

$$\mu\nabla_1^2\psi+ip$$

is an analytic function of $\alpha+i\beta$.

Now (Jeffery, 1921)

$$\begin{aligned} a\nabla_1^2\psi &= \left[(\cosh\alpha-\cos\beta)\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)-2\sinh\alpha\frac{\partial}{\partial\alpha}-2\sin\beta\frac{\partial}{\partial\beta}+(\cosh\alpha+\cos\beta)\right](h\psi) \\ &= 2(A_0+D_0+B_1)+2(B_0+D_1)\alpha+2(2A_1-D_0)\cosh\alpha\cos\beta+2(2C_1-B_0-D_1) \\ &\qquad \sinh\alpha\cos\beta-2(A_1\cosh 2\alpha+C_1\sinh 2\alpha)\cos 2\beta \end{aligned} \tag{12}$$

Therefore we get

$$\begin{aligned} p = \frac{2\mu}{a}[(B_0+D_1)\beta+(2A_1-D_0)\sinh\alpha\sin\beta+(2C_1-B_0-D_1)\cosh\alpha\sin\beta \\ -(A_1\sinh 2\alpha+C_1\cosh 2\alpha)\sin 2\beta] \end{aligned} \tag{13}$$

In conclusion, I wish to express my thanks to Dr. S. Ghosh for guidance and helpful suggestions in course of the work.

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ON THE SLOW STEADY ROTATION OF A SPHERE IN A VISCOUS LIQUID

By

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An approximate solution of the problem of slow steady rotation of a sphere about a diameter in an infinite viscous liquid has been given by Stokes (1845, 1851) by assuming that the liquid particles describe circles round the axis of rotation. When the second order terms are sensible, no steady motion of this type is possible, and Stokes (1845) is of opinion that the sphere acts like a centrifugal fan *i.e* , the motion consists of an inflow at the poles and an outflow at the equator superposed on a motion of rotation. The same picture has also been given by Howarth (1951) in the other extreme case of high Reynold's number.

In this paper, it has been shown that even in the case of slow motion, a solution can be constructed by a process of successive approximations, which gives a picture similar to that envisaged by Stokes.

1. Using spherical polar co-ordinates r, θ, φ, let us suppose that the sphere $r = a$ rotates slowly with angular velocity Ω about the polar axis. If u, v, w be the components of velocity, then omitting the azimuthal variations, the equations of steady motion and the equation of continuity are (Goldstein, 1950)

$$u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \theta}-\frac{v^2+w^2}{r}=-\frac{1}{\varrho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2 u-\frac{2u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial \theta}-\frac{2v\cot\theta}{r^2}\right) \tag{1}$$

$$u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \theta}+\frac{uv}{r}-\frac{w^2\cot\theta}{r}=-\frac{1}{\varrho r}\frac{\partial p}{\partial \theta}+\nu\left(\nabla^2 v+\frac{2}{r^2}\frac{\partial u}{\partial \theta}-\frac{v}{r^2\sin^2\theta}\right) \tag{2}$$

$$u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \theta}+\frac{wu}{r}+\frac{vw\cot\theta}{r}=\nu\left(\nabla^2 w-\frac{w}{r^2\sin^2\theta}\right) \tag{3}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v\sin\theta)=0 \tag{4}$$

On the surface of the sphere $r = a$,

$$u=v=0,\quad w=\Omega a\sin\theta \tag{5}$$

At infinity the velocity vanishes and the pressure is finite.

Stokes' solution of the problem is (Lamb, 1953).

$$u=v=0,\quad w=\frac{\Omega a^3}{r^2}\sin\theta,\quad p=\text{constant}. \tag{6}$$

2. In the second approximation, we take the components of velocity as

$$u,\ v,\quad w = \frac{\Omega a^3}{r^2}\sin\theta + w' \tag{7}$$

Substituting in the left-hand sides of the equations (1)—(3) from the first approximation (6), we get

$$\nu\left(\nabla^2 u - \frac{2u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial\theta} - \frac{2v\cot\theta}{r^2}\right) = \frac{1}{\varrho}\frac{\partial p}{\partial r} - \frac{\Omega^2 a^6}{r^5}\sin^2\theta \tag{8}$$

$$\nu\left(\nabla^2 v + \frac{2}{r^2}\frac{\partial u}{\partial\theta} - \frac{v}{r^2\sin^2\theta}\right) = \frac{1}{\varrho r}\frac{\partial p}{\partial\theta} - \frac{\Omega^2 a^6}{r^5}\sin\theta\cos\theta \tag{9}$$

$$\nabla^2 w' - \frac{w'}{r^2\sin^2\theta} = 0 \tag{10}$$

To satisfy these equations, we assume

$$u = \Omega^2 a^2(3\cos^2\theta - 1)F(r) \tag{11}$$

$$v = \Omega^2 a^2\sin\theta\cos\theta G(r) \tag{12}$$

$$w' = \Omega^2 a^2\sin\theta H(r) \tag{13}$$

$$p = 2\varrho\nu\Omega^2 a^2[f(r) + g(r)\sin^2\theta] \tag{14}$$

Substituting in (4), (8), (9) and (10), we get

$$F'' + \frac{2}{r}F' - \frac{8F}{r^2} - \frac{2G}{r^2} = f' \tag{15}$$

$$G'' + \frac{2}{r}G' - \frac{6G}{r^2} - \frac{12F}{r^2} = \frac{4g}{r} - \frac{a^4}{\nu r^5} \tag{16}$$

$$H'' + \frac{2}{r}H' - \frac{2H}{r^2} = 0 \tag{17}$$

$$F' + \frac{2F}{r} + \frac{G}{r} = 0 \tag{18}$$

$$f' + \frac{2}{3}g' = \frac{a^4}{3\nu r^5} \tag{19}$$

where the dash denotes differentiation with respect to r.

The conditions on the surface of the sphere give

$$F(a) = G(a) = H(a) = 0 \tag{20}$$

Further F, G, H vanish at infinity, and f, g are finite there.

From these equations and boundary conditions, we get

$$F = -\frac{a^3}{8\nu r^2}\left(1-\frac{a}{r}\right)^2, \quad G = \frac{a^4}{4\nu r^3}\left(1-\frac{a}{r}\right), \quad H = 0 \tag{21}$$

$$f = c-\frac{a^3}{4\nu r^3}\left(1-\frac{a}{r}\right), \quad g = \frac{a^3}{6\nu r^3}\left(1-\frac{4a}{3r}\right) \tag{22}$$

3. It is seen from (11) and (21) that u is positive or negative according as $3\cos^2\theta-1 < \text{or} > 0$. Therefore, in the upper hemisphere, there is a radial inflow for $0 \leq \theta < 54°45'$ and a radial outflow for $54°45' < \theta \leq 90°$. when $\theta = 0$, $v = w = 0$, so that there is purely radial inflow at the poles. When $\theta = 90°$, $v = 0$, so that the outflow is in the equatorial plane, but not purely radial.

The only stress component that contributes to the retarding couple is

$$\widehat{r\varphi} = \mu\left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi} + \frac{\partial w}{\partial r} - \frac{w}{r}\right)$$

As u is independent of φ and $w' = 0$, the retarding couple is the same as in Stokes approximation, *viz.*, $8\pi\mu\Omega a^3$.

In conclusion, I wish to express my thanks to Dr. S. Ghosh for guidance and helpful suggestions in course of the work.

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ON THE DISTRIBUTION OF STRESS IN AN INFINITE CIRCULAR CYLINDER OF TRANSVERSELY ISOTROPIC MATERIAL CAUSED BY A BAND OF UNIFORM PRESSURE ON THE BOUNDARY

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When a long cylindrical shaft is acted upon by a band of uniform pressure on the boundary whose width is small compared with the length of the cylinder, the stresses and displacements produced in the cylinder differ considerably from those produced by uniform pressure applied over the whole of the boundary. This problem is of practical importance and can be applied to shrink-fit problems where a short collar is shrunk on a shaft whose length is very great compared with the length of the collar. When the cylinder is isotropic the solution of the problem has been given by Rankin (1944) in the form of a Fourier integral. He has introduced for the purpose the stress function φ (Love, p. 276) which satisfies the differential equation of the fourth order and in terms of which the stresses and the displacements can be expressed. A completely analogous problem from the mathematical point of view is the problem of stress distribution in an infinite solid with a cylindrical hole, when hydrostatic pressure is applied over a small band on the boundary of the hollow. When the material is isotropic an approximate solution of this problem has been given by Westergard (1941). Tranter has also introduced the stress function given by Love. In the present paper the solution of both these problems are given when the material is transversely isotropic with the axis of symmetry at each point parallel to the axis of the cylinder.

Elliott (1948) has shown that in a transversely isotropic material the stresses and displacements can be expressed in terms of two functions which satisfy second order differential equations. In the case of axial-stresses and displacements in terms of these two functions and the elastic moduli. But if the elastic constants are introduced instead of the elastic modulii the expressions for the stresses and displacements become much more simple than those given by Elliott.

1. We introduce cylindrical co-ordinates r, θ, z with the z-axis parallel to the axis of symmetry at each point of the tranversely isotropic material and coinciding with the axis of the cylinder. In a symmetrical deformation about the axis of z we assume

$$u = u(r, z), \quad v = 0, \quad w = w(r, z) \tag{1}$$

The strain components in cylindrical co-ordinates are (Love, *ibid*. p. 56)

$$e_{rr} = \frac{\partial u}{\partial r}, e_{\theta\theta} = \frac{u}{r}, e_{zz} = \frac{\partial w}{\partial z} \tag{2}$$

$$e_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \quad e_{\theta z} = e_{r\theta} = 0$$

The strain energy function for a transversely isotropic material can be written as

$$W = \tfrac{1}{2}c_{11}(e_{rr}^2 + e_{\theta\theta}^2) + \tfrac{1}{2}c_{33}e_{zz}^2 + c_{13}(e_{rr} + e_{\theta\theta})e_{zz} + c_{12}e_{rr}e_{\theta\theta} + \tfrac{1}{2}c_{44}e_{\theta z}^2 + \tfrac{1}{2}c_{44}e_{zr} + \tfrac{1}{2}c_{66}e_{r\theta} \tag{3}$$

where $c_{12} = c_{11} - 2c_{66}$.

The stress components in cylindrical co-ordinates are then

$$\widehat{rr} = c_{11}\frac{\partial u}{\partial r} + c_{12}\frac{u}{r} + c_{13}\frac{\partial w}{\partial z}$$

$$\widehat{\theta\theta} = c_{12}\frac{\partial u}{\partial r} + c_{11}\frac{u}{r} + c_{13}\frac{\partial w}{\partial z} \tag{4}$$

$$\widehat{zz} = c_{13}\frac{\partial u}{\partial r} + c_{13}\frac{u}{r} + c_{33}\frac{\partial w}{\partial z}$$

$$\widehat{zr} = c_{44}\frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right), \quad \widehat{\theta z} = \widehat{r\theta} = 0$$

The stress equations of equilibrium are (Love, *ibid*. p. 90)

$$\frac{\partial \widehat{rr}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{r\theta}}{\partial \theta} + \frac{\widehat{rr} - \widehat{\theta\theta}}{r} = 0$$

$$\frac{\partial \widehat{rz}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{\theta\theta}}{\partial \theta} + \frac{\partial \widehat{\theta z}}{\partial z} + \frac{2\widehat{r\theta}}{r} = 0 \tag{5}$$

$$\frac{\partial \widehat{rz}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{\theta z}}{\partial \theta} + \frac{\partial \widehat{zz}}{\partial z} + \frac{\widehat{zr}}{r} = 0$$

Substituting from (4) in (5) we see that the second of the equations (5) is identically satisfied and the other two reduce to

$$c_{11}\left[\frac{\partial u^2}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right] + c_{44}\frac{\partial^2 u}{\partial z^2} + (c_{13} + c_{44})\frac{\partial^2 w}{\partial r \partial z} = 0 \tag{6}$$

$$(c_{13} + c_{44})\left[\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r}\frac{\partial u}{\partial z}\right] + c_{44}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + c_{33}\frac{\partial^2 w}{\partial z^2} = 0 \tag{7}$$

Assuming that

$$u = \frac{\partial \varphi}{\partial r}, \quad w = k\frac{\partial \varphi}{\partial z} \tag{8}$$

we see that (6) and (7) are satisfied if

$$c_{11}\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}\right)+\{c_{44}+k(c_{13}+c_{44})\}\frac{\partial^2\varphi}{\partial z^2}=0 \tag{9}$$

$$\{(c_{13}+c_{44})+kc_{44}\}\left(\frac{\partial^2\varphi}{\partial r_z}+\frac{1}{r}\frac{\partial\varphi}{\partial r}\right)+kc_{33}\frac{\partial^2\varphi}{\partial z^2}=0 \tag{10}$$

A non-trivial solution of these equations exists only when (9) and (10) are identical, *i.e.*, when

$$\frac{c_{44}+k(c_{13}+c_{44})}{c_{11}}=\frac{kc_{33}}{(c_{13}+c_{44})+kc_{44}}=\nu^2\text{ (say)} \tag{11}$$

Eliminating k between these equations we see that ν^2 satisfies the equation

$$c_{11}c_{44}\nu^4+[c_{13}(c_{13}+2c_{44})-c_{11}c_{33}]\nu^2+c_{33}c_{44}=0 \tag{12}$$

If ν_1^2 and ν_2^2 be the roots of this equation and if φ_1 and φ_2 satisfy the equations

$$\nabla_1^2\varphi+\nu_1^2\frac{\partial^2\varphi_1}{\partial z^2}=0,\ \nabla_1^2\varphi_2+\nu_2\frac{\partial^2\varphi_2}{\partial z^2}=0 \tag{13}$$

they also satisfy the equations (9) and (10).

In terms of φ_1 and φ_2 the displacements and stresses are given by

$$u=\frac{\partial\varphi_1}{\partial r}+\frac{\partial\varphi_2}{\partial r} \tag{14}$$

$$w=k_1\frac{\partial\varphi_1}{\partial z}+k_2\frac{\partial\varphi_2}{\partial z} \tag{15}$$

$$\widehat{rr}=-\frac{c_{11}-c_{12}}{r}\left(\frac{\partial\varphi_1}{\partial r}+\frac{\partial\varphi_2}{\partial r}\right)-c_{44}\left\{(1+k_1)\frac{\partial^3\varphi_1}{\partial z^2}+(1+k_2)\frac{\partial^2\varphi_2}{\partial z^2}\right\} \tag{1n}$$

$$\widehat{\theta\theta}=\frac{c_{11}-c_{12}}{r}\left(\frac{\partial\varphi_1}{\partial r}+\frac{\partial\varphi_2}{\partial r}\right)+\left[(k_1c_{13}-\nu_1^2c_{12}\frac{\partial^2\varphi_1}{\partial z^2}+(k_2c_{13}-\nu_2^2c_{12})\frac{\partial^2\varphi_2}{\partial z^2}\right] \tag{17}$$

$$\widehat{zz}=(k_1c_{33}-\nu_1^2c_{13})\frac{\partial^2\varphi_1}{\partial z^2}+(k_2c_{33}-\nu_2^2c_{13})\frac{\partial^2\varphi_2}{\partial z^2} \tag{18}$$

$$\widehat{zr}=c_{44}\left[(1+k_1)\frac{\partial^2\varphi_1}{\partial r\partial z}+(1+k_2)\frac{\partial^2\varphi_2}{\partial r\partial z}\right] \tag{19}$$

$$\widehat{\theta z}=\widehat{r\theta}=0 \tag{20}$$

where k_1 and k_2 are the values of k corresponding to the values ν_1 and ν_2 of ν.

The formulae for displacement and stresses are analogous to Elliott's formulae (2·4·4) to (2·4·8) and to Shield's (1951) formulae (2·4).

2. The roots ν_1^2 and ν_2^2 of the equation (12) for ν^2 are either real or conjugate imaginaries. Both the cases appear in practical application. For magnesium the roots are real and for zinc they are conjugate imaginaries. Thus we see from (11) that k_1 and k_2 are either real or conjugate imaginaries It is seen from the equations (14) to (20) that for the displacements and stresses to be real the functions φ_1 and φ_2 must be either real of conjugate imaginary. This assumption is consistent with the equations (13). From the form of the solution which will be used in the next section it will be seen that in the application of the above formulae ν_1^2 and ν_2^2 must be positive when they are real. When ν_1^2 and ν_2^2 are real positive we will take ν_1 and ν_2 to be positive and when ν_1^2 and ν_2^2 are conjugate imaginaries we will assume the real parts of ν_1 and ν_2 to be positive.

3. Let us consider an infinite cylinder of radius r with its axis along the axis of z, the material of the cylinder being transversely isotropic with the axis of symmetry parallel to the z-axis. Let a band of the cylindrical surface bounded by $z = \pm h$ be acted upon by a uniform normal thrust P then we have, when $r = a$,

$$\widehat{rr} = -P, \quad \text{when} \quad |z| < h \tag{21}$$

$$\widehat{rr} = 0, \quad \text{when} \quad |z| < h \tag{22}$$

$$\widehat{r\theta} = \widehat{rz} = 0 \quad \text{when} \quad r = a$$

We see that

$$\varphi_1 = I_0(\nu_1\lambda r)\cos\lambda z, \quad \varphi_2 = I_0(\nu_2\lambda r)\cos\lambda z \tag{23}$$

satisfy the equations (13), where I_0 is the modified Bessel's function of the first kind of order zero.

As a solution of the problem we assume

$$\varphi_1 = \int_0^\infty A(\lambda) I_0(\nu_1\lambda r)\cos\lambda z\, d\lambda \tag{24}$$

$$\varphi_2 = \int_0^\infty B(\lambda) I_0(\nu_2\lambda r)\cos\lambda z\, d\lambda \tag{25}$$

Substituting in (14) to (19) and using [Whittaker and Watson, — p. 373]

$$I_0'(z) = I_1(z)$$

we get the displacements and stresses as

$$u = \int_0^\infty [\nu_1 A(\lambda) I_1(\nu_1\lambda r) + \nu_2 B(\lambda) I_1(\nu_2\lambda r)]\lambda\cos\lambda z\, d\lambda \tag{26}$$

$$w = -\int_0^\infty [k_1 A(\lambda) I_0(\nu_1\lambda r) + k_2 B(\lambda) I_0(\nu_2\lambda r)]\lambda\sin\lambda z\, d\lambda \tag{27}$$

$$\widehat{rr} = -\frac{c_{11}-c_{12}}{r}\int_0^\infty [\nu_1 A(\lambda) I_1(\nu_1\lambda r) + \nu_2 B(\lambda) I_1(\nu_2\lambda r)\lambda \cos\lambda z d\lambda + c_{44}\int_0^\infty [1+k_1) A(\lambda) \mathrm{I}_0(\nu_1\lambda r)$$
$$+ (1+k_2) B(\lambda) I_0(\nu_2\lambda r)]\lambda^2 \cos\lambda z d\lambda \qquad (28)$$

$$\widehat{\theta\theta} = \frac{c_{11}-c_{12}}{r}\int_0^\infty [\nu_1 A(\lambda) I_1(\nu_1\lambda r) + \nu_2 B(\lambda) I_1(\nu_2\lambda r)]\lambda \cos\lambda z d\lambda$$
$$-\int_0^\infty [(k_1 c_{33} - \nu_1{}^2 c_{13}) A(\lambda) I_2(\nu_1\lambda r) + (k_2 c_{33} - \nu_2{}^2 c_{13}) B(\lambda) I_0(\nu_2\lambda r)]\lambda^2 \cos\lambda z d\lambda \qquad (29)$$

$$\widehat{zz} = -\int_0^\infty [(k_1 c_{33} - \nu_1{}^2 c_{13}) A(\lambda) I_0(\nu_1\lambda r) + (k_2 c_{33} - \nu_2 c_{13}) B(\lambda) I_0(\nu_2\lambda r)]\lambda^2 \cos\lambda z d\lambda \qquad (30)$$

$$\widehat{zr} = -c_{44}\int_0^\infty [(1+k_1)\nu_1 A(\lambda) I_1(\nu_1\lambda r) + (1+k_2)\nu_2 B(\lambda) I_1(\nu_2\lambda r)]\lambda^2 \sin\lambda z d\lambda \qquad (31)$$

Since $\widehat{rz} = 0$ when $r = a$ we have

$$\frac{A(\lambda)}{(1+k_2)\nu_2 I_1(\nu_2\lambda a)} = \frac{-B(\lambda)}{(1+k_1)\nu_1 I_1(\nu_2\lambda a)} = c(\lambda) \text{ say} \qquad (32)$$

In order that the displacements and stresses may be real we see that $c(\lambda)$ is purely imaginary. On the boundary $r = a$, $\widehat{rr}$ is given by (21). By Fourier's Integral theorem we have

$$(\widehat{rr})_{r=a} = -\frac{2P}{\pi}\int_0^\infty \frac{\sin\lambda h \cos\lambda z}{\lambda} d\lambda \qquad (33)$$

From (28) and (33) we get

$$-\frac{c_{11}-c_{12}}{a}[\nu_1 A(\lambda) I_1(\nu_1\lambda a) + \nu_2 B(\lambda) I_1(\nu_2\lambda a)]\lambda + c_{44}[(1+k_1) A(\lambda) I_0(\nu_1\lambda a)$$
$$+ (1+k_2) B(\lambda) I_0(\nu_2\lambda a)]\lambda^2$$
$$= -\frac{2P}{\pi}\frac{\sin\lambda h}{\pi} \qquad (34)$$

Substituting from (32) we get

$$c(\lambda)\Big[\frac{\nu_1\nu_2}{a}(k_1-k_2)(c_{11}-c_{12}) I_1(\nu_1\lambda a) I_1(\nu_2\lambda a) + c_{44}(1+k_1)(1+k_2)\{\nu_2 I_0(\nu_1\lambda a) I_1(\nu_2\lambda a)$$
$$-\nu_1 I_0(\nu_2\lambda a) I_1(\nu_1\lambda a)\}\Big] = -\frac{2P}{\pi}\frac{\sin\lambda h}{\lambda} \qquad (35)$$

This equation determines $c(\lambda)$ and then $A(\lambda)$ and $B(\lambda)$ are known from (32).

4. For approximate evaluation of the integrals appearing in the equations (26) to (31) in the solution of the problem we split up the interval of integration into two parts $(0, a)$ and (a, ∞), where a is a suitably large number. In the interval (a, ∞) we replace I_0 and I_1 by their asymptotic expansions and calculate the integrals in this range. In the interval $(0, a)$ the integrals are calculated by Filon's method (1928).

5. If instead of a solid circular cylinder we have an infinite solid with a cylindrical hole of circular section and if the material of the solid possesses transverse isotropy with the axis of symmetry parallel to the axis of the hole we have to satisfy in addition to the conditions mentioned in sections 2 and 3 the condition that the displacements and the stresses tend to zero as r tends to infinity. The modified Bessel's functions of the first kind are not suitable for this purpose and we introduce the modified Bessel's functions of the second kind as these functions tend to zero at infinity when the real part of the argument of the function is positive. The modified Bessel's function used here is that given by Whittaker and Watson (1927, p. 373).

The displacements and stresses in the case of the hole is given by

$$u = \int_0^\infty [\nu_1 A(\lambda) K_1(\nu_1\lambda r) + \nu_2 B(\lambda) K_1(\nu_1\lambda r)]\lambda \cos \lambda z d\lambda$$

$$w = -\int_0^\infty [k_1 A(\lambda) K_0(\nu_1\lambda r) + k_2 B(\lambda) K_0(\nu_2\lambda r)]\lambda \sin \lambda z d\lambda$$

$$\widehat{rr} = -\frac{c_{11}-c_{12}}{r} \int_0^\infty [\nu_1 A(\lambda) K_1(\nu_1\lambda r) + \nu_2 B(\lambda) K_1(\nu_2\lambda r)]\lambda \cos \lambda z d\lambda + c_{44}$$
$$\int_0^\infty [(1+k_1) A(\lambda) K_0(\nu_1\lambda r) + (1+k_2) B(\lambda) K_0(\nu_2\lambda r)]\lambda^2 \cos \lambda z d\lambda$$

$$\widehat{\theta\theta} = \frac{c_{11}-c_{12}}{r} \int_0^\infty [\nu_1 A(\lambda) K_1(\nu_1\lambda r) + \nu_2 B(\lambda) K_1(\nu_2\lambda r)]\lambda \cos \lambda z d\lambda$$
$$- \int_0^\infty [(k_1 c_{33} - \nu_1^2 c_{13}) A(\lambda) K_0(\nu_1\lambda r) + (k_2 c_{33} - \nu_2^2 c_{13}) B(\lambda) K_0(\nu_2\lambda r)]\lambda^2 \cos \lambda z d\lambda$$

$$\widehat{zz} = -\int_0^\infty [(k_1 c_{33} - \nu_1^2 c_{13}) A(\lambda) K_0(\nu_1\lambda r) + (k_2 c_{33} - \nu_2^2 c_{13}) B(\lambda) K_0(\nu_2\lambda r)]\lambda^2 \cos \lambda z d\lambda$$

$$\widehat{zr} = -c_{44} \int_0^\infty [(1+k_1)\nu_1 A(\lambda) K_1(\nu_1\lambda r) + (1+k_2)\nu_2 B(\lambda) K_1(\nu_2\lambda r)]\lambda^2 \sin \lambda z d\lambda$$

where

$$\frac{A(\lambda)}{(1+k_2)\nu_2 K_1(\nu_2\lambda a)} = \frac{-B(\lambda)}{(1+k_1)\nu_1 K_1(\nu_1\lambda a)} = c(\lambda)$$

and
$$c(\lambda)\Big[\frac{\nu_1\nu_2}{a}(k_1-k_2)(c_{11}-c_{12})K_1(\nu_1\lambda a)K_1(\nu_2\lambda a)+c_{44}(1+k_1)(1+k_2)\{\nu_2K_0(\nu_1\lambda a)K_1(\nu_2\lambda a)$$
$$-\nu_1K_0(\nu_2\lambda a)K_1(\nu_1\lambda a)\}\lambda\Big]=-\frac{2P}{\pi}\,\frac{\sin\lambda h}{\lambda}.$$

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ON THE DISTRIBUTION OF STRESS IN A HOLLOW AEOLOTROPIC CYLINDER BY A LOCALISED SHEAR ON A BOUNDARY

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The solutions of the problems of elastic distortions of an isotropic cylinder and of an infinite isotropic body with a cylindrical hole due to a localised uniform pressure or a localised uniform shear have been given by Rankin (1944), Tranter (1946) and Das (1953). The present author (1955) has solved the problem of stress distribution in a transversely isotropic cylinder and in an infinite transversely isotropic body with a cylindrical hole due to the action of a band of uniform pressure on the boundary. In the present paper the stress distribution in an aeolotropic cylindrical shell, finite or infinite, is determined when uniform shear in planes perpendicular to the axis acts on a band of a boundary. The material of the cylinder is assumed to possess cylindrical aeolotropy in the most general sense. Transverse isotropy with axis of symmetry parallel to the z-axis and ordinary cylindircal aeolotropy are particular cases of the general cylindrical aeolotropy.

1. We take the z-axis along the axis of the cylindrical shell and use cylindrical co-ordinates r, θ, z. We assume that the material of the cylinder possesses cylindrical aeolotropy in the most general sense, that is, it possesses three planes of symmetry at each point perpendicular to the directions of r, θ, z. The strain energy function of such a material can be written as

$$W = \tfrac{1}{2}c_{11}e_{rr}^2 + \tfrac{1}{2}c_{22}e_{\theta\theta}^2 + \tfrac{1}{2}c_{33}e_{zz}^2 + c_{23}e_{\theta\theta}e_{zz} + c_{31}e_{zz}e_{rr} + c_{12}e_{rr}e_{\theta\theta} + \tfrac{1}{2}c_{44}e_{\theta z}^2 + \tfrac{1}{2}c_{55}e_{zr}^2 + \tfrac{1}{2}c_{66}e_{r\theta}^2 \quad (1)$$

The stress components are given by

$$\widehat{rr} = c_{11}e_{rr} + c_{12}e_{\theta\theta} + c_{13}e_{zz}$$

$$\widehat{\theta\theta} = c_{12}e_{rr} + c_{22}e_{\theta\theta} + c_{23}e_{zz}$$

$$\widehat{zz} = c_{13}e_{rr} + c_{23}e_{\theta\theta} + c_{33}e_{zz}$$

$$\widehat{\theta z} = c_{44}e_{\theta z}, \quad \widehat{zr} = c_{55}e_{zr}, \quad \widehat{r\theta} = c_{66}e_{r\theta}.$$

The components of strain are given by (Love, p. 56)

$$e_{rr} = \frac{\partial u}{\partial \theta}, \; e_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}, \; e_{zz} = \frac{\partial w}{\partial z},$$

$$e_{\theta z} = \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial z}, \; e_{zr} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \quad (3)$$

$$e_{\theta r} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

When the shell is acted upon by shearing stress perpendicular to the axis of the shell on a band over a boundary we assume

$$u = 0, \quad v = v(r, z), \quad w = 0 \tag{4}$$

Then from (3) and (4) we get

$$\begin{gathered} e_{rr} = e_{\theta\theta} = e_{zz} = e_{rz} = 0 \\ e_{\theta z} = \frac{\partial v}{\partial z}, \; e_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} \end{gathered} \tag{5}$$

Then from (2) and (5) get

$$\begin{gathered} \widehat{rr} = \widehat{\theta\theta} = \widehat{zz} = \widehat{rz} = 0 \\ \widehat{\theta z} = c_{44}\frac{\partial v}{\partial z}, \quad \widehat{r\theta} = c_{66}\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) \end{gathered} \tag{6}$$

The stress equations of equilibrium are (Love, p. 90)

$$\begin{gathered} \frac{\partial \widehat{rr}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{\theta r}}{\partial \theta} + \frac{\partial \widehat{rz}}{\partial z} + \frac{\widehat{rr} - \widehat{\theta\theta}}{r} = 0 \\ \frac{\partial \widehat{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{\theta\theta}}{\partial \theta} + \frac{\partial \widehat{\theta z}}{\partial z} + \frac{2\widehat{r\theta}}{r} = 0 \\ \frac{\partial \widehat{rz}}{\partial r} + \frac{1}{r}\frac{\partial \widehat{\theta z}}{\partial \theta} + \frac{\partial \widehat{zz}}{\partial z} + \frac{\widehat{rz}}{r} = 0 \end{gathered} \tag{7}$$

Two of these equations are satisfied by (6) and the third reduces to

$$\frac{\partial^2 v}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{c_{44}}{c_{66}}\frac{\partial^2 v}{\partial z^2} = 0 \tag{8}$$

A solution of (8) independent of z is

$$v = Ar + \frac{B}{r} \tag{9}$$

Let us seek a solution of the type

$$v = Ve^{ikz} \tag{10}$$

where V is a function of r alone.

The equation (8) reduces in this case to

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \left\{(i\varrho)^2 - \frac{1}{r^4}\right\}V = 0 \tag{11}$$

where
$$\varrho^2 = \frac{c_{44}}{c_{66}} k^2 \qquad (12)$$

A solution of (11) is

$$V = AI_1(\varrho r) + Bk_1(\varrho r) \qquad (13)$$

where I_1 and K_1 are modified Bessel's functions of order one of the first and second kinds respectively as defined by Whittaker and Watson (p. 373).

When the cylinder is finite and bounded by the planes $z = \pm L$ and the plane faces are free from tractions we have

$$\widehat{\theta z} = c_{44}\frac{\partial v}{\partial z} = 0 \quad \text{when} \quad z = \pm L \qquad (14)$$

To satisfy this condition we assume

$$v = V \cos kz \qquad (15)$$

where
$$k = n\pi/L \qquad (16)$$

These give
$$\varrho = n\pi/l \qquad (17)$$

where
$$l = L(c_{66}/c_{44})^{\frac{1}{2}}$$

For the solution of our problem we assume

$$v = (A_0 r + B_0/r) + \sum_{n=1}^{\infty}[A_n I_1[(n\pi r/l) + B_n K_1(n\pi r/l)] \cos(n\pi z/L) \qquad (18)$$

This gives

$$\widehat{r\theta} = c_{66}[-2B_0/r^2 + \sum_{n=1}^{\infty} A_n[(n\pi/l)I'_1(n\pi r/l) - (1/r)I_1(n\pi r/l) + B_n[(n\pi/l)K'_1(n\pi r/l)$$
$$-(1/r)K_1(n\pi r/l)]\} \cos(n\pi z/L)] \qquad (19)$$

where the dashes denote differentiation with respect to the argument.

Using the relation [Whittaker and Watson, p. 373]

$$\frac{d}{dz}\{z^{-1}I_1(z) = z^{-1}I_2(z) \qquad (20)$$

we have
$$\widehat{r\theta} = -\frac{2B_0 c_{66}}{r^2} + c_{66}\sum_{n=1}^{\infty}(n\pi/l)[A_n I_2(n\pi r/l) + B_n K_2(n\pi r/l)] \cos(n\pi z/L) \qquad (21)$$

2. If a band of the inner surface $r = a$ bounded by $z = \pm h$ be acted upon by a uniform tangential traction S we have

$$\widehat{r\theta} = -S \quad \text{when} \quad |z| < h \qquad (22)$$

$$\widehat{r\theta} = 0 \quad \text{when} \quad |z| > h \qquad (23)$$

Expanding this boundary value of $\widehat{r\theta}$ in a Fourier's Series of cosines of multiples of $n\pi z/L$ and equating with the value of $\widehat{r\theta}$ on $r=a$ as obtained from (21) we get

$$B_0 = \frac{Sa^2h}{2c_{66}L} \tag{24}$$

and for $n \geq 1$

$$A_nI_2(n\pi a/l)+B_nK_2(n\pi a/l) = -\frac{2Sl}{n^2\pi^2c_{66}}\sin(n\pi h/L) \tag{25}$$

If the boundary $r=b$ be fixed then

$$A_0 = -\frac{Sa^2h}{2c_{66}Lb^2} \tag{26}$$

and for $n \geq 1$

$$A_nI_1\left(\frac{n\pi b}{l}\right)+B_nK_1\left(\frac{n\pi b}{l}\right) = 0 \tag{27}$$

From (25) and (27) we get

$$\frac{A_n}{K_1(n\pi b/l)} = \frac{-B_n}{I_1(n\pi b/l)}$$

$$= \frac{\dfrac{2Sl}{\pi^2n^2c_{66}}\sin(n\pi h/L)}{I_1(n\pi b/l)K_2(n\pi a/l)-I_2(n\pi a/l)K_1(n\pi b/l)} \tag{28}$$

Hence

$$v = -\frac{Sa^2h}{2Lb^2c_{66}}(r-b^2/r)$$

$$-\sum_{n=1}^{\infty}\frac{2Sl}{n^2\pi^2c_{66}}\,\frac{I_1(n\pi b/l)K_1(n\pi r/l)-I_1(n\pi r/l)K_1(n\pi b/l)}{I_1(n\pi b/l)K_2(n\pi a/l)-I_2(n\pi a/l)K_1(n\pi b/b)}\sin(n\pi h/L)\cos(n\pi z/L) \tag{29}$$

$$\widehat{r\theta} = -\frac{Sa^2h}{Lr^2}$$

$$-\sum_{n=1}^{\infty}\frac{2Sl}{n\pi}\,\frac{I_1(n\pi b/l)K_2(n\pi r/l)-I_2(n\pi r/l)K_1(n\pi b/l)}{I_1(n\pi b/l)K_2(n\pi a/l)-I_2(n\pi a/l)K_1(n\pi b/l)}\sin(n\pi h/L)\cos(n\pi z/L) \tag{30}$$

3. By making b tend to infinity we get the case of a cylindrical hole $r=a$ in an infinite transversely isotropic solid subjected to a uniform shear S over a band $-h \leq z \leq h$ of the hole.

Now when $b \to \infty$

$$K_1(n\pi b/l) \to 0 \quad \text{and} \quad I_1(n\pi b/l) \to \infty$$

so that
$$A_n \to 0 \quad \text{and} \quad B_n \to -\frac{2Sl \sin (n\pi h/L)}{n^2\pi^2 c_{66} K_2(n\pi a/L)}$$

We have, therefore, in this case

$$v = \frac{Sa^2h}{2Lc_{66}} \cdot \frac{1}{r} - \sum_{n=1}^{\infty} \frac{2Sl}{n^2\pi^2 c_{66}} \frac{K_1(n\pi r/l)}{K_2(n\pi a/l)} \sin (n\pi h/L) \cos (n\pi z/L) \tag{31}$$

$$\widehat{r\theta} = -\frac{Sa^2h}{L} \cdot \frac{1}{r^2} - \sum_{n=1}^{\infty} \frac{2Sl}{n\pi} \frac{K_2(n\pi r/l)}{K_2(n\pi a/l)} \sin (n\pi h/L) \cos (n\pi z/L) \tag{32}$$

When the medium in isotropic the problem has been solved by Das and our formula (32) agrees with the equation (2·7) of his paper except for a diffence of sign in the second term. This is due to his taking a definition of K function different from that of Whittaker and Watson.

4. Let us next consider the case of an infinite cylindrical shell acting upon by a uniform shear S perpendicular to the axis of a band of the inner boundary.

In this case we assume

$$v = \int_0^{\infty} [A(t)I_1(\alpha tr) + B(t)K_1(\alpha tr)] \cos zt \, dt \tag{33}$$

where
$$\alpha^2 = c_{44}/c_{66}.$$

From (11), (13) and (15) it is seen that v satisfies the equation (8).

Calculating $\widehat{r\theta}$ as before we get

$$\widehat{r\theta} = c_{65}\alpha \int_0^{\infty} t[A(t)I_2(\alpha tr) + B(t)K_2(\alpha tr)] \cos zt \, dt \tag{34}$$

Let on the boundary $r = a$

$$\widehat{r\theta} = -S \quad \text{when} \quad |z| < h$$

$$\widehat{r\theta} = 0 \quad \text{when} \quad |z| > h$$

By Fourier's Integral Theorem we have

$$(\widehat{r\theta})_{r=a} = -\frac{2S}{\pi} \int_0^{\infty} \frac{\sin ht \cos zt}{t} \, dt \tag{35}$$

Comparing the values of $\widehat{r\theta}$ on $r = a$ as given by (34) with that of (35) we get

$$c_{66}\alpha t[A(t)I_2(\alpha ta)+B(t)K_2(\alpha ta)] = -\frac{2S}{\pi}\,\frac{\sin ht}{t} \tag{36}$$

If we assume that the outer boundary $r = b$ is fixed then $v = 0$ when $r = b$ and we have

$$\frac{A(t)}{K_1(\alpha tb)} = \frac{-B(t)}{I_1(\alpha tb)} = \frac{2S}{\pi\alpha c_{66}t^2}\,\frac{\sin ht}{I_1(\alpha tb)K_2(\alpha ta)-I_2(\alpha ta)K_1(\alpha tb)}$$

Hence $$v = -\frac{2S}{\pi\alpha c_{66}}\int_0^\infty \frac{I_1(\alpha tb)K_1(\alpha tr)-I_1(\alpha tr)K_1(\alpha tb)}{I_1(\alpha tb)K_2(\alpha ta)-I_1(\alpha ta)K_1(\alpha tb)}\,\frac{\sin ht\cos zt}{t^2}\,dt$$

$$\widehat{r\theta} = -\frac{2S}{\pi}\int_0^\infty \frac{I_1(\alpha tb)K_2(\alpha tr)-I_2(\alpha tr)K_1(\alpha tb)}{I_1(\alpha tb)K_2(\alpha tb)-I_2(\alpha ta)K_1(\alpha tb)}\,\frac{\sin ht\cos zt}{t}\,dt$$

5. If we make $b\to\infty$ we get the case of an infinite solid with a cylindrical hole acted upon by a band of uniform shear on the hole.

We have $$v = -\frac{2S}{\pi\alpha c_{66}}\int_0^\infty \frac{K_1(\alpha tr)}{K_2(\alpha ta)}\,\frac{\sin ht\cos zt}{t^2}\,dt$$

$$\widehat{r\theta} = -\frac{2S}{\pi}\int_0^\infty \frac{K_2(\alpha tr)}{K_2(\alpha ta)}\,\frac{\sin ht\cos zt}{t}\,dt.$$

The corresponding formulae for isotropic material have been given by Das and our formulae agree with the formula (4·5) of Das except for a difference in sign. This is, as has been mentioned before, due to a definition of K function different from that given by Whittaker and Watson.

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A THEOREM INVOLVING GENERALISED LAPLACE INTEGRAL

By

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1. The object of this paper is to obtain a theorem which involves Laplace and Generalised Laplace Integrals. The Generalised Laplace Integral is given by Dr. R. S. Varma (1951). One of the theorems stated by N. A. Shastri (1944) comes out as a particular case of the theorem stated in this paper.

2. **Theorem.** *If*

$$\psi(p) = p\int_0^\infty e^{-px} f(x)\,dx \tag{2·1}$$

and

$$x^{m-k+\frac{1}{2}} f\left(\frac{1}{x}\right) = x\int_0^\infty e^{-\frac{1}{2}xu}(xu)^{m-\frac{1}{2}} W_{k,m}(xu)\varphi(u)\,du, \tag{2·2}$$

then

$$\psi(p) = 2p^{k+\frac{1}{2}}\int_0^\infty u^m K_{2m}(2(up)^{\frac{1}{2}})\varphi(u)\,du \tag{2·3}$$

provided $R(p) > 0$, $R(\mu+1) > 0$ *and* $R(2m+\mu+1) > 0$ *where* $\varphi(u) = O(u^\mu)$ *for small* u, *and the integral in* (2·3) *is absolutely convergent.*

Proof. Substituting the value of $f(x)$ in (2·1) from (2·2) we get

$$\psi(p) = p\int_0^\infty e^{-px}\left[x^{-k}\int_0^\infty u^{m-\frac{1}{2}} e^{-\frac{1}{2}u/x} W_{k,m}(u/x)\varphi(u)\,du\right]dx \tag{2·4}$$

On changing the order of integration in (2·4)

$$\psi(p) = \int_0^\infty u^{m-\frac{1}{2}}\varphi(u)\left[p\int_0^\infty e^{-px}x^{-k}e^{-\frac{1}{2}u/x}W_{k,m}(u/x)\,dx\right]du$$

Now using the operational representation due to Goldstein (1932)

$$e^{-\frac{1}{2}a/x}W_{k,m}(a/x)x^{-k} \doteqdot 2a^{\frac{1}{2}}p^{k+\frac{1}{2}}K_{2m}(2(ap)^{\frac{1}{2}}) \quad R(a) > 0, \quad R(p) > 0 \tag{2·5}$$

we at once obtain the result (2·3).

The change in the order of integration is justified when the conditions stated in the theorem are satisfied.

3. **Corollary.** In the above theorem put $k = -m + \frac{1}{2}$ and $m = -\frac{1}{2}$, then we obtain the theorem stated by N. A. Shastri (1944):

If
$$\psi(p) \doteqdot f(x)$$

and
$$\frac{1}{p^{s}} f\left(\frac{1}{p}\right) \doteqdot \varphi(x),$$

then
$$\psi(p) = 2p^{\frac{1}{2}s+1} \int_0^\infty t^{-\frac{1}{2}s} K_s(2(pt)^{\frac{1}{2}}) \varphi(t)\, dt$$

provided $R(p) \geq 0$, $\varphi(t)$ *is bounded and integrable and*

$$\int_0^t |\varphi(t)|\, dt$$

s convergent.

4. *Example 1.* Let $\varphi(u) = P_n^{(\alpha,\beta)}(u)$, where $P_n^{(\alpha,\beta)}(u)$ denotes Jacobi's polynomial, then due to Saksena (1952, p. 204)

$$x^{m-k+\frac{1}{2}} f\left(\frac{1}{x}\right) = \frac{1}{n!} \sum_{\nu=0}^{n} \frac{1}{2^\nu . x^\nu} \binom{n}{\nu} (n+\alpha+\beta+1)_\nu (\alpha+\nu+1)_{n-\nu} \frac{\Gamma(2m+\nu+1)\Gamma(\nu+1)}{\Gamma(m-k+\nu+3/2)}$$

$$\times {}_2F_1\left[\begin{matrix} -n+\nu,\ n+\alpha+\beta+\nu+1 \\ \alpha+\nu+1 \end{matrix} ;\ \tfrac{1}{2}\right], \quad (4{\cdot}1)$$

where $R(2m+1) > 0$, $R(x) > 0$.

Hence, after obtaining $\psi(p)$ on term by term interpretation, we get

$$\int_0^\infty u^m K_{2m}(2(up)^{\frac{1}{2}}) P_n^{(\alpha,\beta)}(u)\, du = \frac{1}{2p^{m+1} n!} \sum_{\nu=0}^{n} \frac{1}{2^\nu} \binom{n}{\nu} \frac{(n+\alpha+\beta+1)_\nu}{p^\nu} (n+\alpha+\beta+1)_\nu$$

$$\times (\alpha+\nu+1)_{n-\nu} \Gamma(2m+\nu+1)\Gamma(\nu+1)\ {}_2F_1\left[\begin{matrix} -n+\nu,\ n+\alpha+\beta+\nu+1 \\ \alpha+\nu+1 \end{matrix} ;\ \tfrac{1}{2}\right]$$

where $R(2m+1) > 0$, $R(p) > 0$ and $R(\nu+m-k+\frac{3}{2}) > 0$. (4·2)

Since $P_n^{(\alpha,\beta)}(u)$ is a very general type of function, the result obtained in this example is of very general nature.

Example 2. Let $\varphi(u) = J_\mu(2au^{\frac{1}{2}}) J_\nu(2au^{\frac{1}{2}})$ then due to Saksena (1952, p. 206)

$$x^{m-k+\frac{1}{2}} f\left(\frac{1}{x}\right) = \frac{a^{\mu+\nu}\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+1)\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+2m+1)}{x^{\frac{1}{2}(\mu+\nu)}\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\frac{1}{2}(\mu+\nu)+m-k+\frac{3}{2})}$$

$$\times {}_4F_4\left[\begin{matrix} \frac{1}{2}(\mu+\nu)+1,\ \frac{1}{2}(\mu+\nu)+1,\ \frac{1}{2}(\mu+\nu)+2m+1,\ \frac{1}{2}(\mu+\nu+1) \\ \mu+1,\ \nu+1,\ \mu+\nu+1,\ \frac{1}{2}(\mu+\nu)+m-k+\frac{3}{2} \end{matrix} ;\ -\frac{4a^2}{x}\right] \quad (4{\cdot}3)$$

where $R(\frac{1}{2}(\mu+\nu)+1) > 0$, $R(\frac{1}{2}(\mu+\nu)+2m+1) > 0$, $R(x) > 0$.

Hence, after obtaining $\psi(p)$ on term by term interpretation, we get

$$2p^{k+\frac{1}{2}}\int_0^\infty u^m K_{2m}(2(up)^{\frac{1}{2}})J_\mu(2au^{\frac{1}{2}})J_\nu(2au^{\frac{1}{2}})du = \frac{a^{\mu+\nu}\Gamma(\frac{1}{2}(\mu+\nu)+1)\Gamma(\frac{1}{2}(\mu+\nu)+2m+1)}{p^{m-k+\frac{1}{2}\mu+\frac{1}{2}\nu+\frac{1}{2}}\Gamma(\mu+1)\Gamma(\nu+1)}$$

$$\times {}_4F_3\left[\begin{matrix}\frac{1}{2}(\mu+\nu)+1,\ \frac{1}{2}(\mu+\nu)+1,\ \frac{1}{2}(\mu+\nu)+2m+1,\ \frac{1}{2}(\mu+\nu+1) \\ \mu+1,\ \nu+1,\ \mu+\nu+1\end{matrix}\ ;\ -\frac{4a^2}{p}\right] \tag{4·4}$$

where
$$R(\tfrac{1}{2}(\mu+\nu+1)) > 0,\quad R(\tfrac{1}{2}(\mu+\nu)+2m+1) > 0,$$
$$R(p) > 4a^2 \quad\text{and}\quad R(m-k+\tfrac{1}{2}(\mu+\nu+\tfrac{3}{2})) > 0.$$

Example 3. Let $\varphi(u) = u^n$, then using Goldstein's integral (1932)

$$x^{m-k+\frac{1}{2}}f\left(\frac{1}{x}\right) = \frac{\Gamma(n+2m+1)\Gamma(n+1)}{\Gamma(n+m-k+3/2)}\frac{1}{x^n}$$

where $R(n+2m+1) > 0$, $R(n+1) > 0$.

Also
$$\psi(p) = \frac{\Gamma(n+2m+1)\Gamma(n+1)}{p^{n+m-k+\frac{1}{2}}}$$

$$R(p) > 0,\ R(n+2m+1) > 0,\ R(n+1) > 0 \quad\text{and}\quad R(n+m-k+\tfrac{3}{2}) \geq 0.$$

Hence
$$\int_0^\infty u^{m+n}K_{2m}(2(up)^{\frac{1}{2}})du = \frac{\Gamma(n+2m+1)\Gamma(n+1)}{2p^{n+m+1}}$$

$$R(p) > 0,\ R(n+2m+1) > 0 \quad\text{and}\quad R(n+1) > 0.$$

Now on substituting $2(up)^{\frac{1}{2}} = t$, $m = \frac{1}{2}\nu$ and $n = \frac{1}{2}(\mu-\nu)-1$, we get

$$\int_0^\infty t^{\mu-1}K_\nu(t)dt = 2^{\mu-2}\Gamma\left(\frac{\mu+\nu}{2}\right)\Gamma\left(\frac{\mu-\nu}{2}\right)$$

$$R(\tfrac{1}{2}\mu+\tfrac{1}{2}\nu) > 0 \quad\text{and}\quad R(\tfrac{1}{2}\mu-\tfrac{1}{2}\nu) > 0, \tag{4·5}$$

a result given in Watson's Bessel Functions (1952, p. 388).

Example 4. Let $\varphi(u) = e^{-u}u^\nu$

then due to C. B. Rathie (1952, p. 232)

$$x^{m-k+\frac{1}{2}}f\left(\frac{1}{x}\right) = \frac{\Gamma(\nu+1)\Gamma(\nu+2m+1)}{\Gamma(\nu+m-k+\frac{3}{2})x^\nu}{}_2F_1\left(\nu+1,\ \nu+2m+1\,;\ \nu+mk+\tfrac{3}{2}\,;\ -\frac{1}{x}\right)$$

$$R(\nu) > -1 \quad\text{and}\quad R(\nu+2m) > -1. \tag{4·6}$$

Now using the integral of C. S. Meijer (1939)

$$W_{k,m}(z) = \frac{1}{\Gamma(c)}e^{-z/2}z^k\int_0^\infty e^{-t}t^{c-1}\,{}_2F_1\left(\tfrac{1}{2}-k+m,\ \tfrac{1}{2}-k-m\,;\ c\,;-\frac{t}{z}\right)dt$$

$$R(c) > 0,\ |\arg z| < \pi,\ z \neq 0. \tag{4·7}$$

we obtain
$$\psi(p) = \frac{\Gamma(\nu+1)\Gamma(\nu+2m+1)}{p^{-k}} e^{\frac{1}{2}p} W_{-\nu-m-\frac{1}{2}, m}(p)$$

$$= 2p^{k+\frac{1}{2}} \int_0^\infty u^{m+\nu} e^{-u} K_{2m}(2(up)^{\frac{1}{2}}) du$$

which finally gives

$$2x^{m+\nu} K_{2m}(2x^{\frac{1}{2}}) \doteqdot \Gamma(\nu+1)\Gamma(\nu+2m+1) W_{-\nu-m-\frac{1}{2}, m}\left(\frac{1}{p}\right) e^{1/(2p)} \frac{1}{p^{m+\nu-\frac{1}{2}}} \tag{4·8}$$

where
$$R(\nu+1) > 0, \quad R(\nu+2m+1) > 0,$$
$$R(p) > 0 \quad \text{and} \quad R(\nu+m-k+\tfrac{3}{2}) > 0.$$

This result is otherwise given by Shastri (1944).

Example 5. Let $\varphi(u) = u^{\nu-1} {}_rF_s(\alpha_1, \alpha_2, \ldots\ldots \alpha_r; \beta_1, \ldots\ldots \beta_s; \pm u)$
then due to Rathie (1952)

$$x^{m-k+\frac{1}{2}} f\left(\frac{1}{x}\right) = \frac{\Gamma(\nu)\Gamma(\nu+2m)}{\Gamma(\nu+m-k+\frac{1}{2}) x^{\nu-1}} {}_{r+2}F_{s+1}\left(\begin{matrix} \alpha_1, \ldots\ldots \alpha_r, \nu, \nu+2m \\ \beta_1, \ldots\ldots \beta_s, \nu+m-k+\frac{1}{2} \end{matrix}; \pm\frac{1}{x}\right).$$

$R(\nu) \geq 0$, $R(\nu+2m) > 0$, $R(p) > 0$ when $r < s$; if $r = s$ then $R(p) > 1$.

Hence, after obtaining $\psi(p)$ on term by interpretation, we get

$$\frac{2p^{\nu+m}}{\Gamma(\nu)\Gamma(\nu+2m)} \int_0^\infty u^{m+\nu-1} K_{2m}(2(up)^{\frac{1}{2}} {}_rF_s\left(\begin{matrix} \alpha_1, \ldots\ldots \alpha_r; \pm u \\ \beta_1, \ldots\ldots \beta_s \end{matrix}\right) du$$

$$= {}_{r+2}F_s\left(\begin{matrix} \alpha_1, \ldots\ldots \alpha_r, \nu, \nu+2m \\ \beta_1, \ldots\ldots \beta_s \end{matrix}; \pm\frac{1}{p}\right) \tag{4·9}$$

$R(\nu+m-k+\frac{1}{2}) > 0$, $R(\nu) > 0$, $R(p) > 0$, $R(\nu+2m) > 0$ and $r+2 < s$; when $r+1 = s$, then $R(p) > 1$. This result is given by S. Sinha (1944, p. 28)

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STRESS DISTRIBUTION IN A THIN PARABOLIC PLATE DUE TO A CONCENTRATED FORCE AT THE FOCUS*

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Summary. The stresses in a thin plate of elastic material with a parabolic boundary, due to a concentrated force acting at the focus parallel to the axis of the parabola, have been found by the method of superposition. A method of determining the stresses *directly* without the help of the stress function or the stress potential has been found very helpful. The displacements are found by the use of complex variable. Numerical values of the edge-stress have also been considered.

1. **Introduction.** The problem of a semi-infinite elastic solid with a plane boundary which is commonly known as the "problem of Boussinesq and Cerruti" has been the object of numerous researches. The natural extension of this problem is that in which the plane boundary is replaced by a curved one. The case of a parabolic boundary is of some practical importance Conrad (1949), by the use of complex potential, calculated the stresses and displacements in a parabolic cylinder due to the hydrostatic pressure on the boundary, taking the problem as one of plane strain. Tiffen (1952) reduced the parabolic boundary to a straight boundary by the conformal mapping. It was shown by the author (Paria, 1952) that the stresses in similar cases can be directly obtaind by a method which will be used in this paper also. In all these cases, either the surface tractions or the surface displacements were prescribed on a certain region near the vertex of the parabolic boundary. Here we propose to find the stresses and displacements when a nucleus of strain of a certain type acts at a point inside the body, *e.g.*, a concentrated force acting at the focus parallel to the axis of the parabola.

2. **Method of solution.** Assuming that the plate is in a state of generalised plane stress the average stresses $\overset{\smile}{\xi\xi}$, $\overset{\smile}{\xi\eta}$ and $\overset{\smile}{\eta\eta}$ in two-dimensional orthogonal curvilinear coordinates (ξ, η) can be expressed as [Sen (1948)]

$$\frac{8\overset{\smile}{\xi\xi}}{h^2} = \frac{\partial r^2}{\partial\eta}\frac{\partial\odot}{\partial\eta} - \frac{\partial r^2}{\partial\xi}\frac{\partial\odot}{\partial\xi} + \frac{4\odot}{h^4} + F \tag{1}$$

$$\frac{8\overset{\smile}{\eta\eta}}{h^2} = \frac{\partial r^2}{\partial\xi}\frac{\partial\odot}{\partial\xi} - \frac{\partial r^2}{\partial\eta}\frac{\partial\odot}{\partial\eta} + \frac{4\odot}{h^2} - F \tag{2}$$

$$\frac{8\overset{\smile}{\xi\eta}}{h^2} = -\frac{\partial r^2}{\partial\eta}\frac{\partial\odot}{\partial\xi} - \frac{\partial r^2}{\partial\xi}\frac{\partial\odot}{\partial\eta} - G \tag{3}$$

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where $r^2 = x^2+y^2$, (x, y) being the cartesian co-ordinates, $\frac{1}{h^2} = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2$ and

$\odot = \overset{\smile}{xx} + \overset{\smile}{yy} = \overset{\smile}{\xi\xi} + \overset{\smile}{\eta\eta}$ is a plane harmonic function satisfying,

$$\nabla_1^2 \odot = h^2\left[\frac{\partial^2 \odot}{\partial \xi^2} + \frac{\partial^2 \odot}{\partial \eta^2}\right] = 0 \tag{4}$$

Also, F and G are two conjugate functions of (ξ, η) such that

$$F+iG = f(x+iy), \text{ a function of } \xi+i\eta$$

The stress components $\overset{\smile}{\xi\xi}$, $\overset{\smile}{\eta\eta}$ and $\overset{\smile}{\xi\eta}$ are given in terms of $\overset{\smile}{xx}$, $\overset{\smile}{xy}$ and $\overset{\smile}{yy}$ in cartesian co-ordinates by the relations

$$\overset{\smile}{\xi\xi} = \cos^2\theta\, \overset{\smile}{xx} + \sin^2\theta\, \overset{\smile}{yy} + 2\sin\theta\cos\theta\, \overset{\smile}{xy} \tag{5}$$

$$\overset{\smile}{\eta\eta} = \sin^2\theta\, \overset{\smile}{xx} + \cos^2\theta\, \overset{\smile}{yy} - 2\sin\theta\cos\theta\, \overset{\smile}{xy} \tag{6}$$

$$\overset{\smile}{\xi\eta} = -\sin\theta\cos\theta\,(\overset{\smile}{xx} - \overset{\smile}{yy}) + (\cos^2\theta - \sin^2\theta)\, \overset{\smile}{xy} \tag{7}$$

where θ is the angle between the normal to the curve ξ = constant and the x-axis and is given by the relations

$$\cos\theta = h\frac{\partial y}{\partial \eta} = h\frac{\partial x}{\partial \xi}, \quad \sin\theta = h\frac{\partial y}{\partial \xi} = -h\frac{\partial x}{\partial \eta} \tag{8}$$

If the stresses $\overset{\smile}{\xi\xi}$ and $\overset{\smile}{\xi\eta}$ are prescribed on the boundary ξ = constant, we assume the plane harmonic function $\odot$ involving certain unknown constants and determine the surface value of G from (8). The functions F and G being conjugate to each other, both F and G are known at every point on the plane. Then, from (1), the unknown constants in $\odot$ are determined. Thus the function $\odot$, F and G being known, the stresses are known by the relations (1)—(8).

To find the components of displacements we make use of the complex variable (Love, 1952, pp. 214-5). Since $\odot$ has already been determined, Δ, the dilatation is given by the relation

$$\odot = 2(\lambda'+\mu)\Delta \tag{9}$$

where μ is the modulus of rigidity and λ' is one of Lamé's constants as considered in generalised plane stress. Also, the dilatation Δ and rotation ϖ are such that, the expression $(\lambda'+2\mu)\Delta + 2i\mu\varpi$ is a function of $z = x+iy$ i.e., a function of $\zeta = \xi+i\eta$. Since Δ is known, ϖ is also known. Now, if we introduce a new function $L+iM$ of $x+iy$ by means of the equation

$$L+iM = \int\{(\lambda'+2\mu)\Delta + 2i\mu\varpi\}d(x+iy) \tag{10}$$

the components of displacements (u, v) parallel to the x and y axes are given by

$$u = \frac{L}{2\mu} + \frac{\lambda' + \mu}{2\mu(\lambda' + 2\mu)}\, y\, \frac{\partial L}{\partial y} \tag{11}$$

$$v = \frac{M}{2(\lambda' + 2\mu)} - \frac{\lambda' + \mu}{2\mu(\lambda' + 2\mu)}\, y\, \frac{\partial M}{\partial y} \tag{12}$$

We can, of course, add other suitable functions to u and v to make them single-valued.

If the components of stresses are required in cartesian co-ordinates, they can be found from the relations (9), (11) and (12) together with the relations

$$\widetilde{xx} = \lambda'\Delta + 2\mu\frac{\partial u}{\partial x} \tag{13}$$

$$\widetilde{yy} = \lambda'\Delta + 8\mu\frac{\partial v}{\partial y} \tag{14}$$

$$\widehat{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{15}$$

3. Solution for parabolic Boundary. The transformation

$$z = \zeta^2 \tag{16}$$

or

$$x + iy = (\xi + i\eta)^2 \tag{16.1}$$

gives,

$$x = \xi^2 - \eta^2, \quad y = 2\xi\eta \tag{17}$$

and

$$r = \xi^2 + \eta^2, \quad 1/h^2 = 4(\xi^2 + \eta^2)^2 \tag{18}$$

It is easily seen that $\xi = \alpha$ (a constant) gives a parabola having x-axis as the axis of the parabola, vertex at $x = \alpha^2$, $y = 0$, focus at $x = y = 0$ and the concavity turned towards the negetive direction of the x axis. (Fig. 2)

If we suppose that the plate extends to infinity in all directions, the stress components due to a concentrated force P acting at the origin parallel to the x-axis in the positive direction are given by (Love, *loc. cit.*, p. 209, with changed notation)

$$\widetilde{xx}_1 = \frac{Q}{4}\frac{x}{r^2}\left[\frac{2y^2}{r^2} - (3 - 2\bar{\sigma})\right] \tag{19}$$

$$\widetilde{yy}_1 = \frac{Q}{4}\frac{x}{r^2}\left[(1 - 2\bar{\sigma}) - \frac{2y^2}{r^2}\right] \tag{20}$$

$$\widetilde{xy}_1 = -\frac{Q}{4}\frac{y}{r^2}\left[(1 - 2\bar{\sigma}) + \frac{2x^2}{r^2}\right] \tag{21}$$

where $\bar{\sigma}$ is the modified Poisson ratio defined by $\bar{\sigma} = \dfrac{\lambda'}{2(\lambda' + \mu)}$.

and
$$Q = \frac{P}{\pi(1-\bar{\sigma})} \tag{22}$$

In (ξ, η) co ordinates, these stress camponents can be written as, using the relations (5)-(7),

$$\breve{\xi\xi}_1 = -\frac{Q}{4}\left[\frac{3\xi^4+\eta^4}{(\xi^2+\eta^2)^3} - \frac{2\bar{\sigma}}{\xi^2+\eta^2}\right] \tag{23}$$

$$\breve{\eta\eta}_1 = \frac{Q}{4}\left[\frac{\xi^4+3\eta^4}{(\xi^2+\eta^2)3} - \frac{2\bar{\sigma}}{\xi^2+\eta^2}\right] \tag{24}$$

$$\breve{\xi\eta}_1 = -\frac{Q}{2}\,\frac{\xi\eta(\xi^2-\eta^2)}{(\xi^2+\eta^2)3} \tag{25}$$

On $\xi = \alpha$

$$(\breve{\xi\xi}_1)_{\xi=\alpha} = -\frac{Q}{4}\left[\frac{3\alpha^4+\eta^4}{(\alpha^2+\eta^2)^3} - \frac{2\bar{\sigma}}{\alpha^2+\eta^2}\right]$$

$$(\breve{\xi\eta}_1)_{\xi=\alpha} = -\frac{Q\alpha}{2}\,\frac{\eta(\alpha^2-\eta^2)}{(\alpha^2+\eta^2)^3}$$

which can be expressed in the integral form as,

$$(\breve{\xi\xi}_1)_{\xi=\alpha} = -\frac{Q}{8\alpha}\int_0^\infty (m^2\alpha^4+m\alpha+3-4\sigma)e^{-m\alpha}\cos m\eta\,dm \tag{26}$$

$$(\xi\eta_1)_{\xi=\alpha} = -\frac{Q}{8}\int_0^\infty m(m\alpha-1)e^{-m\alpha}\sin m\eta\,dm \tag{27}$$

In order to cancel the stresses (26) and (27) on the boundary, a second stress system $\breve{\xi\xi}_2$, $\breve{\eta\eta}_2$ and $\breve{\xi\eta}_2$ is to be superposed in such a manner that on $\xi = \alpha$

$$(\breve{\xi\xi}_2)_{\xi=\alpha} = -(\breve{\xi\xi}_1)_{\xi=\alpha} \tag{28}$$

$$(\breve{\xi\eta}_2)_{\xi=\alpha} = -(\breve{\xi\eta}_1)_{\xi=\alpha} \tag{29}$$

To find the second syetem, we assume, as a solution of the equations (1)—(3),

$$\odot_2 = \int_0^\infty A(m)e^{m\xi}\cos m\eta\,dm \tag{30}$$

where $A(m)$ is a function of m.

Substituting in (3) and using the value of $(\breve{\xi\eta}_2)_{\xi=\alpha}$ from (29) and (27), we have.

$$(G)_{\xi=\alpha} = -4\int_0^\infty (\alpha^2+\eta^2)mAe^{m\alpha}(\eta\cos\eta-\alpha\sin m\eta)dm \tag{31}$$

The above value of G will be satisfied if we take

$$F+iG = 4\int_0^\infty \zeta(2\alpha-\zeta)^2 mAe^{m\zeta}dm-4Q\int_0^\infty \zeta(2\alpha-\zeta)m(m\alpha-1)e^{-m(2\alpha-\zeta)}dm \tag{32}$$

Hence,

$$(F)_{\xi=\alpha} = 4\int_0^\infty (\alpha^2+\eta^2)mAe^{m\alpha}(\alpha\cos m\eta+\eta\sin m\eta)dm-4Q\times$$

$$\times\int_0^\infty (\alpha^2+\eta^2)m(m\alpha-1)e^{-m\alpha}\cos m\eta\, md \tag{33}$$

The relation (1) is satisfied for $\xi=\alpha$ with the values of $(\overset{\smile}{\xi\xi}_2)_{\xi=\alpha}$, $\odot_2$ and $(F)_{\xi=\alpha}$ as given by (28), (30) and (33) respectively, provided,

$$A(m) = \frac{Q}{4\alpha}(2m^2\alpha^2+3-4\bar{\sigma})e^{-2m\alpha} \tag{34}$$

Substitution of the value of $A(m)$ from (34) in (30) gives, after integration,

$$\odot_2 = \frac{Q}{4\alpha}\frac{\xi'}{\xi'^2+\eta^2}\left[\frac{4\alpha^2(\xi'^2-3\eta^2)}{(\xi'^2+\eta^2)^2}+3-4\bar{\sigma}\right] \tag{35}$$

where

$$\xi' = 2\alpha-\xi \tag{36}$$

Also, from (32) and (34), $F+iG = \dfrac{Q\zeta}{\alpha}\left[\dfrac{4\alpha^2}{(2\alpha-\xi)2}+\dfrac{4\alpha}{2\alpha-\zeta}+3-4\bar{\sigma}\right]$, (32·1)

The values of $\odot_2$, F and G being known from (32) and (35) the stress components $\overset{\smile}{\xi\xi}_2$, $\overset{\smile}{\eta\eta}_2$ and $\overset{\smile}{\xi\eta}_2$ can be calculated from the relations (1)—(3). The complete stress components $\overset{\smile}{\xi\xi}$, $\overset{\smile}{\eta\eta}$ and $\overset{\smile}{\xi\eta}$ are then given by

$$\xi\xi = \xi\xi_1+\xi\xi_2, \quad \text{etc.}$$

In the present discussion, we shall calculate the actual values of the stress components in cartesian co-ordinates by the method already pointed out.

4. Determination of displacements. Corresponding to the superposed stress system $\overset{\smile}{\xi\xi}_2$, $\overset{\smile}{\eta\eta}_2$ and $\overset{\smile}{\xi\eta}_2$, let u_2 and v_2 be the components of displacements parallel to the x and y axes and Δ_2 and ϖ_2 be the dilatation and rotation. Then, from (9) and (30),

$$(\lambda'+2\mu)\Delta_2 = K\int_0^\infty A(m)e^{m\xi}\cos m\eta\, dm \tag{37}$$

where

$$K = \frac{\lambda'+2\mu}{2(\lambda'+\mu)} = 1-\bar{\sigma} \tag{38}$$

Since $(\lambda'+2\mu)\Delta_2+2i\mu\varpi_2$ is a function of $z=x+iy$ *i.e.* of $\zeta=\xi+i\eta$, the relation (37) will be satisfied ff

$$(\lambda'+2\mu)\Delta_2+2i\mu\varpi_2 = K\int_0^\infty A(m)e^{m\zeta}dm \tag{39}$$

Substituting the value of $A(m)$ from (54) and evaluating the integral and remembering that $z = \zeta^2$ we get

$$(\lambda'+2\mu)\Delta_2+2i\mu\varpi_2 = \frac{KQ}{2\sigma}\left[\frac{4\alpha^2}{(2\alpha-z^{\frac{1}{2}})^3}+\frac{3-4\bar{\sigma}}{2\alpha-z^{\frac{1}{2}}}\right] \tag{39·1}$$

The relation (10) gives

$$L_2+iM_2 = \frac{KQ}{2\alpha}\left[\frac{4\alpha^2(z^{\frac{1}{2}}-\alpha)}{(2\alpha-z^{\frac{1}{2}})^2} -(3-4\bar{\sigma})\{z^{\frac{1}{2}}+2\alpha\log(2\alpha-z^{\frac{1}{2}})\}\right] \tag{40}$$

Hence,

$$L_2 = \frac{KQ}{2\alpha}\left[\frac{4\alpha^2}{\xi'^2-\eta^2}\left\{\frac{\alpha(\xi'^2-\eta^2}{\xi'^2+\eta^2}-\xi'\right\}-(3-4\bar{\sigma})\{2\alpha-\xi'+\alpha\log(\xi'^2+\eta^2)\}\right] \tag{41}$$

$$M_2 = \frac{KQ}{2^{\alpha}}\left[\frac{4\alpha^2\eta}{\xi'^2+\eta^2}\left\{\frac{2\alpha\xi'}{\xi'+\eta^2}-1\right\}-(3-4\bar{\sigma})\left\{\eta-2\alpha\tan^{-1}\frac{\eta}{\xi'}\right\}\right]$$

From (11) and (12)

$$\frac{16\mu\alpha}{Q}u_2 = 4K\left[\frac{4\alpha^2}{\xi'^4+\eta^2}\left\{\frac{\alpha(\xi'^2-\eta^2)}{\xi'^2+\eta^2}-\xi'\right\}-(3-4\bar{\sigma})\{2\alpha-\xi'+\alpha\log(\xi'^2+\eta^2)\}\right]$$
$$+\frac{\eta y}{\xi'^2+\eta^2}\left[\frac{4\alpha^2(\eta^2-3\xi'^2)}{(\xi'^2+\eta^2)^2}-(3-4\bar{\sigma})\right] \tag{42}$$

$$\frac{16\mu\alpha}{Q}v_2 = 4(1-2\bar{\sigma})\left[\frac{4\alpha^2\eta}{\xi'^2+\eta^2}\left\{\frac{2\alpha\xi'}{\xi'^2+\eta^2}-1\right\}-(3-4\bar{\sigma})\left\{\eta-2\tan^{-1}\frac{\eta}{\xi'}\right\}\right]$$
$$-\frac{\xi' y}{\xi'^2+\eta^2}\left[\frac{4\alpha^2(\xi'^2-3\eta^2)}{(\xi'^2+\eta^2)^2}+(3-4\bar{\sigma})\right] \tag{43}$$

The components of displacement (u_1, v_1) corresponding to the stress system $\overset{\smile}{xx}_1$, $\overset{\smile}{yy}_1$ and $\overset{\smile}{xy}_1$ are given by Love (loc. cit., p. 209, with changed notation)

$$\left.\begin{aligned}\frac{8\mu}{Q}u_1 &= -\left[(3-4\bar{\sigma})\log r+\frac{y^2}{r^2}\right]\\ \frac{8\mu}{Q}v_1 &= \frac{xy}{r^2}.\end{aligned}\right\} \tag{44}$$

Hence the components of total displacement (u, v) are given by

$$u = u_1+u_2, \quad v = v_1+v_2.$$

6. Determination of the stresses. From (9) and (35), we have,

$$\Delta_2 = \frac{Q}{8\alpha(\lambda'+\mu)}\frac{\xi'}{\xi'^2+\eta^2}\left[\frac{4\alpha^2(\xi'^2-3\eta^2)}{(\xi'^2+\eta^2)^2}+3-4\bar{\sigma}\right] \tag{45}$$

The relation (11) gives

$$e_{xx} = \frac{\partial u_2}{\partial x} = \frac{1}{2\mu}\frac{\partial M}{\partial y} + \frac{1}{4\mu K} y \frac{\partial^2 M}{\partial y^2} \tag{46}$$

From (13) and (15) with the help of (41), or, (40)

$$\frac{8\alpha}{Q}\overset{\smile}{xx}_2 = \frac{2\xi'}{\xi'^2+\eta^2}\left[\frac{4\alpha^2(\xi'^2-3\eta^2)}{(\xi'^2+\eta^2)^2} + 3 - 4\bar{\sigma}\right]$$

$$- \frac{y\eta}{(2\alpha-\xi')^2+\eta^2}\frac{1}{(\xi'^2+\eta^2)^2}\left[\frac{12\alpha^2}{(\xi'^2+\eta^2)^2}\{4\xi'(2\alpha-\xi')(\xi'^2-\eta^2)\right.$$

$$\left. - (\xi'^4+\eta^4-6\xi'^2\eta^2)\} + (3-4\bar{\sigma})\{2\xi'(2\alpha-\xi')-(\xi'^2-\eta^2)\}\right] \tag{47}$$

$$\frac{8\alpha}{Q}\overset{\smile}{xy}_2 = \frac{\eta}{\xi'^2+\eta^2}\left[\frac{4\alpha^2(\eta^2-3\xi'^2)}{(\xi'^2+\eta^2)^2} - (3-4\bar{\sigma})\right] - \frac{y}{(2\alpha-\xi')^2+\eta^2}\frac{1}{(\xi'^2+\eta^2)^2}$$

$$\times\left[\frac{12\alpha^2}{(\xi'^2+\eta^2)^2}\{(2\alpha-\xi')(\xi'^4+\eta^4-6\xi'^2\eta^2)+4\xi'\eta^2(\xi'^2-\eta^2)\}\right.$$

$$\left. + (3-4\bar{\sigma})\{(2\alpha-\xi')(\xi'^2-\eta^2)+2\xi'\eta^2\}\right] \tag{48}$$

Also, $$\overset{\smile}{yy}_2 = \odot_2 - \overset{\smile}{xx}_2 \tag{49}$$

where $\odot_2$ is given by (35)

The complete stress systems $\overset{\smile}{xx}$, $\overset{\smile}{xy}$, $\overset{\smile}{yy}$ are given by $\overset{\smile}{xx} = \overset{\smile}{xx}_1 + \overset{\smile}{xx}_2$. etc.

6. Calculation of the edge stress. From (23) and (24),

$$\odot_1 = \overset{\smile}{\xi\xi}_1 + \overset{\smile}{\eta\eta}_1 = -\frac{Q}{2}\frac{\xi^2-\eta^2}{(\xi^2+\eta^2)^2}$$

Hence $$\odot = \odot_1 + \odot_2 = \frac{Q}{4}\left[\frac{\xi'}{\alpha(\xi'^2+\eta^2)}\left\{\frac{4\alpha^2(\xi'^2-3\eta^2)}{(\xi'^2+\eta^2)^2} + 3 - 4\bar{\sigma}\right\} - \frac{2(\xi^2-\eta^2)}{(\xi^2+\eta^2)^2}\right] \tag{50}$$

On the edge $\xi = \alpha,$

$$(\overset{\smile}{\eta\eta})_{\xi=\alpha} = (\odot)_{\xi=\alpha} = \frac{Q}{4}\left[\frac{2(\alpha^4-6\alpha^2\eta+\eta^4}{(\alpha^4+\eta^2)^3} + \frac{3-4\bar{\sigma}}{\alpha^2+\eta^2}\right]$$

If we put $$\eta = \alpha t \tag{51}$$

The above relation reduces to

$$(\overset{\smile}{\eta\eta})_{\xi=\alpha} = \frac{Q}{4\alpha^4}\left[\frac{2(1-6t^2+t^4)}{(1+t^2)^3} + \frac{3-4\bar{\sigma}}{1+t^2}\right] \tag{52}$$

In the particular case if we take $\sigma = \frac{1}{4}$ *i.e.*, $\bar{\sigma} = \frac{1}{5}$ the relation (52) together with (22) gives

$$(\widetilde{\eta\eta})_{\xi=\alpha} = \frac{P}{16\pi\alpha^2}\, I(t) \tag{53}$$

where

$$I(t) = \frac{21(1+t^4)-88t^2}{(1+t^2)^3} \tag{54}$$

The values of $I(t)$ for different values of t are given in Table I while the continuous curve in Figure I gaves the graphical representation of $I(t)$.

7. **Comparison with the straight boundary.** We draw the tangent at the vertex of the parabola (Fig. 2). Let P' be a point on the tangent line corresponding to a point P on the parabola. If the plate were semi-infinite with the tangent line as the straight boundary, the edge stress due to a concentrated force P will be given by Sneddon (1946, p. 412, with necessary modification).

$$\widetilde{yy}_e = \frac{2P\alpha^2}{\pi}\,\frac{\alpha^4-\bar{\sigma}y^2/(1-\bar{\sigma})}{(\alpha^4+y^2)^2} \tag{55}$$

where $\widetilde{yy}_e$ denotes the edge stress on the straight boundary and y is one of the co-ordinates of P'.

If we take

$$y = 2\alpha^2 t \tag{56}$$

in consistency with the relation (51) and put $\bar{\sigma} = \frac{1}{5}$ the relation (55) reduces to

$$\widetilde{yy}_e = \frac{P}{16\pi\alpha^2}\, I'(t) \tag{57}$$

where

$$I'(t) = \frac{32(1-t^2)}{(1+4t^2)^2} \tag{58}$$

The values of $I'(t)$ for different values of t are given in Table I, while the broken line in Fig. I, gives the graphical representation of $I'(t)$.

It may be noticed that $I(t)$ is less than $I'(t)$ for the values of t ranging from 0·0 to 0·4 approximately, while between the range 0·4 and 1·0, the former is greater than the latter. At $t = 1·0$, $I'(t)$ takes the zero value, reaches the minimum at $t = 1·5$ and then increases and tends to zero. But $I(t)$ has a minimum when t is slightly greater than 0·8, has a maximum at $t \sim 2·5$ and then tends to zero, retaining its positive value throughout the whole range.

TABLE—I.

t	0·0	0·1	0·2	0·3	0·4	0·45	0·5	0·6
$I(t)$	21·0	20·01	17·34	13·71	9·90	8·24	6·56	3·99
$I'(t)$	32·0	29 29	22·83	15·74	9·99	7·90	6·00	3·44

t	0·8	1·0	1·5	2 0	3·0	4·0	5·0
$I(t)$	1·20	0·50	—	1 64	1·38	0·97	0·69
$I'(t)$	0·91	0·0	—0·40	—0·33	—0·19	—0·11	—0·08

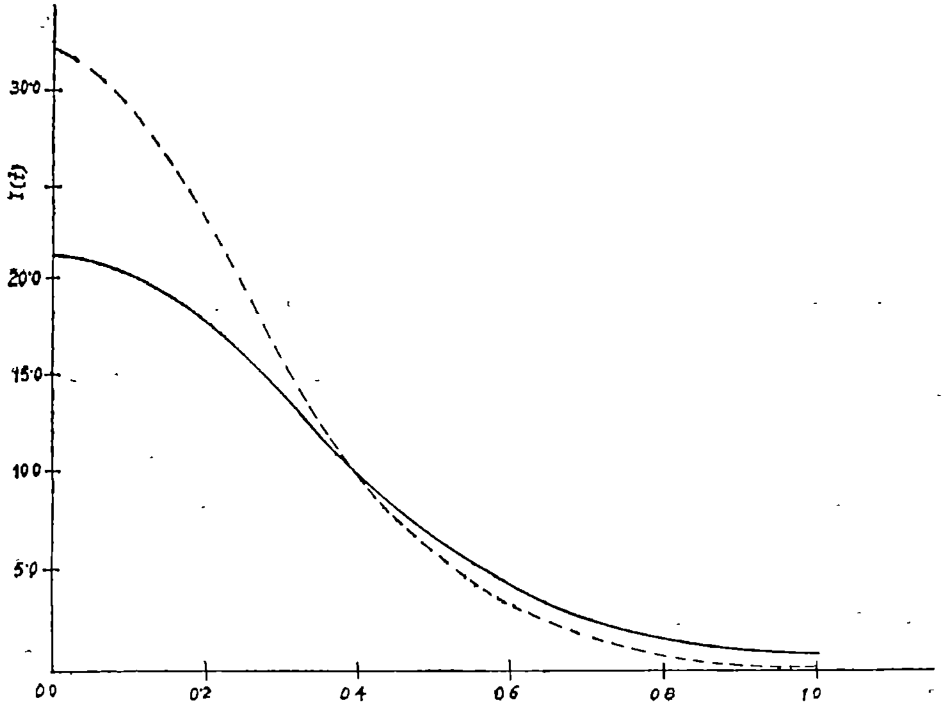
I(t)
30·0
20·0
15·0
10·0
5·0
0·0
0·2
0·4
0·6
0·8
1·0

Fig. 1

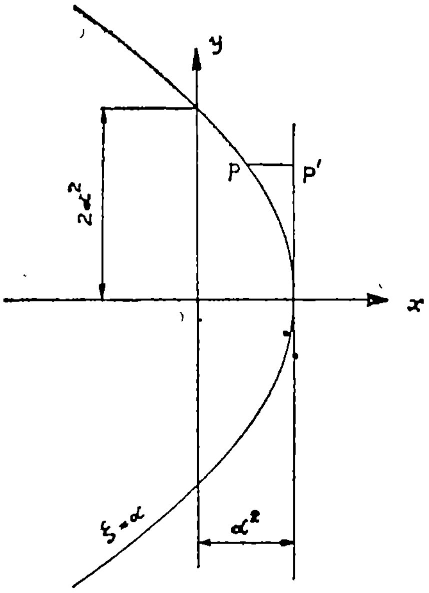
y
x
P
P'
$2\alpha^2$
α^2
$\xi=\alpha$

Fig. 2

8. Stress components in curvilinear co-ordinates. From (32·1) by seperating real and imaginary parts, we get

$$F = \frac{Q}{\alpha}\left[\frac{4\alpha^2}{(\xi'^2+\eta^2)^2}\{\xi(\xi'^2-\eta^2)-2\xi'\eta^2\}+\frac{4\alpha(\xi\xi'-\eta^2)}{\xi'^2+\eta^2}+(3-4\bar{\sigma})\xi\right] \tag{59}$$

$$G = \frac{Q\eta}{\alpha}\left[\frac{4\alpha^2(\xi'^2-\eta^2+2\xi\xi')}{(\xi'^2+\eta^2)}+\frac{4\alpha(\xi+\xi')}{\xi'^2+\eta^2}+3-4\bar{\sigma}\right] \tag{60}$$

Then, from (1)—(3) with the help of (35), (59) and (60), we have,

$$\begin{aligned}\frac{32\alpha}{Q}\overset{\smile}{\xi\xi}{}^2 = {} & \frac{4\xi'}{(\xi'^2+\eta^2)^3}\{4\alpha^2(\xi'^2-3\eta^2)+(3-4\bar{\sigma})(\xi'^2+\eta^2)^2\}-(\xi'^2+\eta^2)^{-4}[2\xi'\eta^2\{24\alpha^2(\xi'^2-\eta^2)\\ & +(3-4\bar{\sigma})(\xi'^2+\eta^2)^2\}+\xi\{12\alpha^2(\xi'^4+\eta^4-6\xi'^2\eta^2)+(3-4\bar{\sigma})(\xi'^4-\eta^4)(\xi'^2+\eta^2)\}]\\ & +(\xi^2+\eta^2)^{-1}(\xi'^2+\eta^2)^{-2}[4\alpha^2\{\xi(\xi'^2-\eta^2)-2\xi'\eta^2\}+4\alpha(\xi\xi'-\eta^2)(\xi'^2+\eta^2)\\ & +(3-4\bar{\sigma})\xi(\xi'^2+\eta^2)^2]\end{aligned} \tag{61}$$

$$\begin{aligned}\frac{32\alpha}{Q}\overset{\smile}{\xi\eta}_2 = {} & \eta(\xi'^2+\eta^2)^{-4}[2\xi\xi'\{24\alpha^2(\xi'^2-\eta^2)+(3-4\bar{\sigma})(\xi'^2+\eta^2)^2-\{12\alpha^2(\xi'^4+\eta^4-6\xi'^2\eta^2)\\ & +(3-4\bar{\sigma})(\xi'^4-\eta^4)(\xi'^2+\eta^2)\}]-\eta(\xi^2+\eta^2)^{-1}(\xi^2+\eta^2)^{-2}[4\alpha^2(\xi'^2-\eta^2+2\xi\xi')\\ & +4\alpha(\xi+\xi')(\xi'^2+\eta^2)+(3-4\bar{\sigma})(\xi'^2+\eta^2)^2]\end{aligned} \tag{6\ }$$

$$\eta\eta_2 = \odot_2-\xi\xi_2 \tag{63}$$

In conclusion, I thank Dr. B. R. Seth for his kind suggestion of the problem.

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A GENERALIZATION OF THE RIEMANN-LEBESGUE THEOREM

By

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1. A theorem and its extended form. A principal result contained in this paper is the following

Theorem. *Let $K(x, y)$ be bounded measurable on the plane region $-\infty<x<\infty$, $0\leq y<\omega$ and be periodic with period ω in y. Let $K(x, \lambda x)$ be measurable on $(-\infty, \infty)$ for all large λ. Then for any $f(x) \in L(-\infty, \infty)$, we have*

$$\lim_{\lambda\to\infty}\int_{-\infty}^{\infty} f(x)K(x, \lambda x)\, dx = \frac{1}{\omega}\int_{-\infty}^{\infty} f(x)\, dx \int_{0}^{\omega} K(x, y)\, dy. \tag{1}$$

It is clear that this result involves the following Reimann-Lebesgue theorem [Kestelman (1937), p. 169] as a particular case : *If $\varphi(x)$ is bounded measurable over $(-\infty, \infty)$ and if there is a positive constant δ such that $\varphi(x+\delta) = -\varphi(x)$ for every x in $(-\infty, \infty)$, then for any $f(x) \in L(-\infty, \infty)$ we have*

$$\lim_{\lambda\to\infty}\int_{-\infty}^{\infty} f(x)\varphi(\lambda x)\, dx = 0. \tag{2}$$

In fact, this is easily deduced from our theorem by taking $K(x, y) = \varphi(y)$ and $\omega = 2\delta$ so that

$$\int_{0}^{\omega} K(x, y)\, dy = \int_{0}^{\delta} \varphi(y)\, dy + \int_{\delta}^{2\delta} \varphi(y)\, dy = 0.$$

The theorem may also be stated in a more general form. Let us denote

$$(x) = (x_1, \dots, x_n) \quad (\lambda x) = (\lambda_1 x_1, \dots, \lambda_n x_n), \quad d(x) = d(x_1, \dots, x_n),$$
$$(0) = (0, \dots, 0), \quad (\infty) = (\infty, \dots, \infty).$$

Similarly, we shall use the abbreviation $\int_{(0)}^{(\omega)}$ for the symbol $\int_{0}^{\omega_1}\cdots\int_{0}^{\omega_n}$ of the n-tuple integral. Then the general form of the theorem reads as follows:

Let R_n be the euclidean n-space. Let W be a $2n$ dimensional domain consisting of all points $(x_1, \dots, x_n; y_1, \dots, y_n)$ such that $(x) \in R_n$, $0\leq y_i\leq\omega_i$, $(i = 1, \dots, n)$. Let $K((x), (y))$ and $f((x))$ be real-valued functions satisfying the conditions:

- (i) *$K((x), (y))$ is bounded measurable over W and is periodic in each y_i with period ω_i $(i = 1, 2, \dots, n)$.*

(*ii*) *$K((x), (\lambda x))$ is measurable over R_n for all large $\lambda_1, \ldots, \lambda_n$.*

(*iii*) *$f((x))$ is summable over R_n.*

Then the following formula is valid

$$\lim_{(\lambda)\to(\infty)} \int_{R_n} f((x))K((x), (\lambda x))\, d(x) = \frac{1}{\omega_1 \ldots \omega_n} \int_{R_n} f((x))\, d(x) \int_{(0)}^{(\omega)} K((x), (y))\, d(y), \tag{3}$$

where $(\lambda)\to(\infty)$ means that all λ_i's are simultaneously and independently tending to $+\infty$.

The formula (3) can be established without difficulty by means of (1) and by appealing to the general Fubini theorem [McShane (1944), pp. 137-147]. In what follows we shall give a detailed proof of the case $n = 1$.

2. Proof of the theorem. In order to prove our theorem, we shall make use of a result due to Maréchal and Wilkins [See Maréchal (1947), Wilkins (1949), Grosswald (1951), Brunk and Ewing (1953)]:

Lemma. *If $f(r, \theta)$ is continuous on $0 \leq r \leq R$, $0 \leq \theta \leq 2\pi$ and is periodic with period 2π in θ, then*

$$\lim_{a\to 0+} 2\pi a \int_C f(r, \theta)\, ds = \int_0^{2\pi} \int_0^R f(r, \theta)\, r\, dr\, d\theta, \tag{4}$$

where the line integral on the left-hand side is extended over a portion of an archmidean spiral included in the circle $r = R$, viz.

C: $r = a\theta, \quad (0 \leq r \leq R,\ 0 \leq \theta < +\infty).$

Evidently, the left-hand side is of (4) is equivalent to

$$\lim_{a\to 0+} 2\pi \int_0^R f(r, r/a)(a^2 + r^2)^{\frac{1}{2}}\, dr = \lim_{\lambda\to\infty} 2\pi \int_0^R f(r, \lambda r)\, r\, dr.$$

By change of variables, $r = x$, $\theta = (2\pi/\omega)y$, we may write

$$\lim_{\lambda\to\infty} \int_0^X f_1(x, \lambda x)x\, dx = \frac{1}{\omega} \int_0^X dx \int_0^\omega f_1(x, y)x\, dy, \tag{4 bis}$$

where $f_1(x, y) = f(r, \theta)$, so that $f_1(x, y)x \to 0$ $(x \to 0)$.

Suppose that $F(x, y)$ is a continuous function defined on $0 \leq x \leq X$, $0 \leq y \leq \omega$ and is periodic with period ω in y. Then for any given small number $\eta > 0$, we can choose a positive number $\delta = \delta(\eta) < X$ such that, for all $\lambda > 0$,

$$\left| \int_0^\delta F(x, \lambda x)\, dx \right| < \eta, \quad \left| \frac{1}{\omega} \int_0^\delta dx \int_0^\omega F(x, y)\, dy \right| < \eta.$$

On the region $\delta \leq x \leq X$, $0 \leq y \leq \omega$ we may define $f_1(x, y) = F(x, y)/x$, so that $f_1(x, y)$ is continuous and bounded over the region, and moreover $F(x, y) = f_1(x, y)x$. Hence (4 *bis*) implies

$$\lim_{\lambda\to\infty} \int_\delta^X F(x, \lambda x) = \frac{1}{\omega} \int_\delta^X dx \int_0^\omega F(x, y)\, dy,$$

A comparison with the preceding two inequalities obviously leads to

$$\left| \lim_{\lambda\to\infty} \int_0^X F(x, \lambda x)\, dx - \frac{1}{\omega} \int_0^X dx \int_0^\omega F(x, y)\, dy \right| < 2\eta.$$

This is equivalent to

$$\lim_{\lambda\to\infty} \int_0^X F(x, \lambda x)\, dx = \frac{1}{\omega} \int_0^X dx \int_0^\omega F(x, y)\, dy. \tag{4$'$}$$

More generally, if $H(x, y)$ is bounded and continuous on $0 \leq y < \omega$ and is periodic with period ω in y, if $g(x)$ is a continuous function, absolutely integrable over $0 \leq x < \infty$, then we have

$$\lim_{\lambda\to\infty} \int_0^\infty g(x) H(x, \lambda x)\, dx = \frac{1}{\omega} \int_0^\infty g(x)\, dx \int_0^\omega H(x, y)\, dy \tag{5}$$

In fact, since $\int_0^\infty |g(x)|\, dx < -\infty$ and* $\sup |H(x, y)| < +\infty$, we see that the convergence of $\int_0^\infty gH\, dx$ is uniform with regard to all $\lambda > 0$, and consequently,

$$\lim_{\lambda\to\infty} \lim_{X\to\infty} \int_0^X gH\, dx = \lim_{X\to\infty} \lim_{\lambda\to\infty} \int_0^X gH\, dx = \frac{1}{\omega} \int_0^\infty g\, dx \int_0^\omega H\, dy$$

The next step is to show that if $f(x)$ is Lebesgue integrable over $(0, \infty)$ and if $H(x, y)$ is defined as before, then

$$\lim_{\lambda\to\infty} \int_0^\infty f(x)\, H(x, \lambda x)\, dx = \frac{1}{\omega} \int_0^\infty f(x)\, dx \int_0^\omega H(x, y)\, dx \tag{6}$$

Let $\varepsilon > 0$ be freely chosen. Since $f(x)$ can be approximated by a continuous function $g(x)$ (absolutely integrable over $(0, \infty)$) such that

$$\int_0^\infty |f(x) - g(x)|\, dx < \varepsilon,$$

it follows immediately that

$$\overline{\lim_{\lambda\to\infty}} \left| \int_0^\infty f(x)\, H(x, \lambda x)\, dx - \frac{1}{\omega} \int_0^\infty f(x)\, dx \int_0^\omega H(x, y)\, dy \right|$$

$$\leq \overline{\lim_{\lambda\to\infty}} \left\{ \left| \int_0^\infty [f(x) - g(x)]\, H(x, \lambda x)\, dx \right| + \left| \frac{1}{\omega} \int_0^\infty [g(x) - f(x)]\, dx \int_0^\omega H(x, y)\, dy \right| \right\}$$

$$< \varepsilon \cdot \sup |H(x, y)| + \frac{\varepsilon}{\omega} \sup_x \left| \int_0^\omega H(x, y)\, dy \right| = A\varepsilon,$$

where A is an absolute constant, so that (6) is obtained by letting $\varepsilon \to 0+$.

---

* Here $\sup |H(x, y)|$ denotes the supremum of $|H(x, y)|$ on the plane region $0 \leq x < \infty$, $0 \leq y \leq \omega$.

By hypothesis of the theorem, $K(x, \lambda x)$ is bounded and measurable on $[0, \infty)$ for every large fixed λ (say $\lambda > \lambda_0$). Consequently $|f(x)K(x, \lambda x)|$ is integrable over $[0, \infty)$. Let $\varepsilon > 0$ be given arbitrarily and denote $M_\lambda = \sup_x |K(x, \lambda x)|$. Then by the absolute continuity of a Lebesgue integral, there is a $\delta > 0$ such that

$$M_\lambda \int_{E_0} |f(x)| \, dx < \varepsilon \tag{7}$$

whenever E_0 is a measurable set contained in $[0, \infty)$ with measure $mE_0 < \delta$. By Lusin's theorem, there exists a continuous function $g_\lambda(x)$ defined on $[0, \infty)$ such that

$$mE(g_\lambda(x) \neq K(x, \lambda x)) < \delta, \quad \sup_x |g_\lambda(x)| \leq M_\lambda.$$

Let us now take $E_0 = E(g_\lambda(x) \neq K(x, \lambda x))$. Then from (7) we obtain

$$\left| \int_{E_0} f(x)[K(x, \lambda x) - g_\lambda(x)] dx \right| < 2\varepsilon.$$

Hence it follows that

$$\left| \int_0^\infty f(x)[K(x, \lambda x) - g_\lambda(x)] \, dx \right| < 2\varepsilon. \tag{8}$$

Similarly, by means of Lusin's theorem in the 2-dimensional case [McShane (1944), pp. 236-237] and by the absolute continuity of a double integral, we have a bounded integral, we have a bounded continuous function $H_1(x, y)$ defined on the plane region $0 \leq x < \infty$, $0 \leq y \leq \omega$ such that

$$\left| \int_0^\infty f(x) dx \int_0^\omega K(x, y) \, dy - \int_0^\infty f(x) \, dx \int_0^\omega H_1(x, y) \, dy \right| < 2\varepsilon. \tag{9}$$

Here $H_1(x, 0) = H_1(x, \omega)$ and the definition of $H_1(x, y)$ in the whole region $0 \leq x < \infty$, $0 \leq y < \infty$ can be completed by the periodicity of y, that is, we may define $H_1(x, y + k\omega) = H_1(x, y)$ for $k = 1, 2, \ldots\ldots$, $(0 \leq y \leq \omega)$. Note that line segments are of 2-dimensional measure zero and that the value of a double integral is unaltered when its integrand changes value on any set of measure zero. Hence by the absolute continuity of an integral, we can also choose the bounded continuous function $H_1(x, y)$ to be identical with $g_\lambda(x)$ along the line $y = \lambda x$ with (9) still holding.

$$\left| \int_0^\infty f(x)[K(x, \lambda x) - H_1(x, \lambda x)] dx \right| < 2\,\varepsilon.$$

Moreover, by means of (6), we me may choose $\lambda_1 > \lambda_0$ so large that the following inequality

$$\left| \int_0^\infty f(x) H_1(x, \lambda x \, dx - \frac{1}{\omega} \int_0^\infty f(x) dx \int_0^\omega H_1(x, y) dy \right| < \varepsilon \tag{10}$$

holds whenever $\lambda > \lambda_1$. Thus by comparing (8) with (10) and (9), we find for all $\lambda > \lambda_1$,

$$\left| \int_0^\infty f(x) K(x, \lambda x) dx - \frac{1}{\omega} \int_0^\infty f(x) dx \int_0^\omega K(x, y) dy \right| < 3\varepsilon + \frac{2}{\omega} \varepsilon.$$

Consequently we have

$$\lim_{\lambda\to\infty}\int_0^\infty f(x)K(x,\lambda x)=\frac{1}{\omega}\int_0^\infty f(x)dx\int_0^\omega K(x, y)dy.$$

The case $\int_{-\infty}^{0}$ can be treated in exactly the same manner. Hence our theorem is proved.

As is easily seen, it is possible that a 2-dimensional measurable function $K(x, y)$ is not linear measurable for $y=\lambda x$. Hence the condition concerning the measurability of $K(x, \lambda x)$ for all large λ cannot be omitted in our theorem.

3. Application. As an immediate consequence of the theorem, we have

Corollary. *Let $K(x, y)$ by bounded and continuous on $-\infty<x<\infty$, $0\leq y\leq\omega$ and be periodic with period ω in y. Then the formula (1) holds for any $f(x)$ belonging to $L(-\infty, \infty)$.*

In particular, if we take $K(x, y)=F(\cos y, \sin y)$ and assume

$$\Phi(z)=F(\tfrac{1}{2}(z+z^{-1}), \frac{1}{2i}(z-z^{-1}))$$

to be an analytic function of z on $|z|\leq 1$, the formula (1) yields

$$\lim_{\lambda\to\infty}\int_{-\infty}^{\infty} f(x)F(\cos\lambda x, \sin\lambda x)dx=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)dx\int_0^{2\pi} F(\cos y, \sin y)dy$$

$$=\frac{1}{2\pi i}\left(\int_{-\infty}^{\infty} f(x)dx\right)\left(\int_{\text{circuit } |z|=1}\Phi(z)z^{-1}dz\right). \tag{11}$$

This formula can be used for the evaluation of certain limits, *e.g*,

$$\text{(i)}\qquad \lim_{\lambda\to\infty}\int_a^b \frac{\cos^2(3\lambda x)}{1-2p\cos(2\lambda x)+p^2}f(x)dx=\tfrac{1}{2}\left(\frac{1-p+p^2}{1-p}\right)\int_a^b f(x)dx,$$

$$\text{(ii)}\qquad \lim_{\lambda\to\infty}\int_a^b f(x)e^{\cos\lambda x}\cos(n\lambda x-\sin\lambda x)dx=\frac{1}{n!}\int_a^b f(x)dx,$$

where $f(x)\in L(a, b)$, $0<p<1$ and n is any positive integer.

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ANHARMONIC PULSATIONS OF TWO-PHASE HOMOGENEOUS MODEL

By

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Abstract. In this paper, the anharmonic oscillations of a two-phase homogeneous model, consisting of a homogeneous core surrounded by an envelope of uniform density, have been studied. The core radius is taken to be half of the total radius of the model and the core density is taken to be 10 times the envelope density. For this model the period of oscillation for the fundamental mode is about double (in fact 2.18) the period of the first mode. Because in this case we expect considerable skewness. The problem is solved using numerical method and it is found that for surface amplitude equal to .05, the skewness in the velocity-time curve is 1.48 and the increase in the period is 1.2% of the period for the fundamental mode for vanishing amplitude.

1. Rosseland (1943) developed the general theory of anharmonic pulsations in order to explain the observed skewness in the velocity-time curve for Cepheid variables. Subsequently Bhatnagar and Kothari (1944) obtained for homogeneous model, period and skewness for any value of amplitude taking the ratio of specific heats $\nu = 5/3$. C. Prasad (1949) studied the anharmonic oscillations of homogeneous and inverse square model and Kushwaha and Bhatnagar (1951) for Roche Model retaining second order terms in the eq. of motion. The skewness as expected, due to the lack of resonance between the fundamental and the higher mode, is appreciably low.

In the present note we shall consider the two-phase model of a homogenous compressible core, surrounded by an envelope of small unifom density introduced by Bhatnagar (1945) with a view to study the effect of central condensation on skewness and period. However, he worked out the problem strictly for uniform oscillations. Recently C. Prasad (1953) has studied the general oscillation problem for this model for vanishing amplitude. In order to see how the resonance between the fundamental and the first mode affects the skewness, we take the ratio of core density to envelope density ten and the ratio of the core radius to the total radius of the model one half, for which the ratio of the period of the fundamental mode to that of the first mode is 2. 18. After calculating the amplitude and its derivative more accurately than given by C. Prasad, we proceed to study the anharmonic pulsations by the method of numerical integration.

2. **Equations of motion.** Assuming the oscillations to be adiabatic the equation of motion is

$$\ddot{r} = -\frac{r^2}{\varrho_0 r_0^2} \frac{\partial}{\partial r_0}\left[p_0\left(\frac{r_0^2}{r^2 r'}\right)^{\nu}\right] - g_0\left(\frac{r_0}{r}\right)^2 \tag{1}$$

where r denotes the distance from the centre, ϱ_0, p_0, g_0 and r_0 are the equilibrium values of density, pressure, gravity and distance from the centre, γ the ratio of specific heats; dashes denote differentiation with respect to r_0

If now we write

$$r = r_o(1+\eta)$$

which gives

$$\ddot{r} = r_o\ddot{\eta} \text{ and } r' = 1+\eta+r_o\eta'$$

equation (1) becomes for finite amplitude

$$\varrho_o r_o\ddot{\eta} = -(1+\eta)^2 \frac{\partial}{\partial r_o} [p_o(1+\eta)^{-2\gamma}(1+\eta+r_o\eta')^{-\gamma}] - g_o\varrho_o(1+\eta)^{-2} \tag{2}$$

As suggested by Rosseland we expand η in the form of the series

$$\eta = \sum_k \eta_k(r_o)\; q_k(t) \tag{3}$$

where η_k's are the orthogonal eigenfunctions corresponding to the eigen-values $2\pi/\sigma_k$ for the harmonic oscillations and vanishing amplitude case and q_k's are unknown functions of time which are to be determined from equation (2). We normalize η_k's to unity at the surface of the star so that the displacement at any time at the surface will be given by

$$\eta = \sum_k q_k(t) \tag{4}$$

We substitute η from (2) in (3), expand the right hand side in power of q_k's and neglect the powers of q higher than the second considering the oscillations to be small. On multiplying the resulting equation by $r_o^3\,\eta_k$ and integrating from the centre $r_o = 0$ to the boundary $r_o = R$ of the star we obtain

$$\frac{d^2 q_k}{d\tau^2} + \beta_k q_k = \frac{\gamma}{\sigma_1^{\,2} I_k}[\sum_i D_{ijk}\, q_i^2 + 2\sum_{ij} D_{ijk}\, q_i\, q_j] \tag{5}$$

where

$$\tau = \sigma_1\, t \tag{6}$$

$$\beta_k = \frac{\sigma_k^{\,2}}{\sigma_1^{\,2}} \tag{7}$$

and

$$I_k = \int_0^R \varrho_o\, r_o^4 \eta_k^{\,2} dr_o \tag{8}$$

$$\begin{aligned} D_{ijk} = {} & -\tfrac{1}{2}\left(3-\frac{4}{\gamma}\right)(3\gamma+1)\int_0^R p'_o r_o^3 \eta_i\eta_j\eta_k\, dr_o \\ & +\tfrac{1}{2}(3\gamma-1)\int_0^R p_o r_o^4(\eta_i\eta'_j\eta'_k+\eta'_i\eta_j\eta'_k+\eta'_i\eta'_j\eta_k)\, dr_o \\ & +\tfrac{1}{2}(\gamma+1)\int_0^R p_o r_o^5\, \eta'_i\eta'_j\eta'_k\, dr_o \end{aligned} \tag{9}$$

where i, j, k take all positive integral values.

The valus of q_k's as solutions of equation (5) are obtained in the form of Fourier Series

$$\left.\begin{aligned} q_1 &= a_0+a_1\cos n\tau+a_2\cos 2n\tau+a_3\cos 3n\tau+\dots\dots\dots \\ q_2 &= b_0+b_1\cos n\tau+b_2\cos 2n\tau+b_3\cos 3n\tau+\dots\dots\dots \\ q_3 &= c_0+c_1\cos n\tau+c_2\cos 2n\tau+c_3\cos 3n\tau+\dots\dots\dots \end{aligned}\right\} \tag{10}$$

Thus we get the constants a_0, a_1, a_3, ...; b_0, b_1, b_3, b_3...... and c_0, c_1, c_2, c_3,...... in terms of one constant say a_1, and there by η as function of time.

3. The two-phase homogeneous model. We take a model consisting of a compressible core of uniform density ϱ, surrounded by an envelope of uniform density σ, and suppose the core extending to bR the radius of the whole star being R.

The differential equation of small adiabatic radial oscillation of a star

$$\frac{d^2\eta_k}{dr_0^2}+\frac{4-\mu}{r_0}\,\frac{d\eta_k}{dr_0}+\left(\frac{\sigma_k^2\varrho_0}{rp_0}-\frac{\alpha\mu}{r_0^2}\right)\eta_k=0 \tag{11}$$

where $\mu = g_0\varrho_0r_0/p_0$

and $\alpha = 3-4/r.$

reduces for the core and the envelope to the form

$$(k'^2-x^2)\frac{d^2\eta_k}{dx^2}+\frac{4k'^2-6x^2}{x}\,\frac{d\eta_k}{dx}+J_k\eta_k=0 \tag{12}$$

and

$$\{k'+(1-k')x-x^3\}\frac{d^2\eta_k}{dx^2}+\frac{3k'+4(1-k')x-6x^3}{x}\,\frac{d\eta_k}{dx}+(J'_kx^3-k'\alpha)\frac{\eta_k}{x^3}=0 \tag{13}$$

where $x = r_0/R$

$$k = 2\left(\frac{\varrho-\sigma}{\sigma}\right)b^3$$

$$k' = b^2+\left(\frac{\sigma}{\varrho}\right)^2\left\{1-b^2+k'\left(\frac{1}{b}-1\right)\right\}$$

$$J_k = \frac{3\sigma_k}{2\pi G\gamma\varrho}-2\alpha$$

$$J'_k = 1\sigma_k^2/2\pi G\gamma\sigma-2\alpha$$

The x solution of (12) satisfying the requisite boundary conditions at the centre o the star can be expressed in terms of hypergeometric function:

$$\eta = A\ {}_2F_2\left[\frac{5}{4}+\tfrac{1}{2}\left(\frac{25}{4}+J_k\right)^{\frac{1}{2}},\quad \frac{5}{4}-\tfrac{1}{2}\left(\frac{25}{4}+J_k\right)^{\frac{1}{2}};\quad \frac{5}{2};\quad \frac{x^2}{k'^2}\right]$$

The value of σ_k is so adjusted such that η_k and η'_k/η_k are continuous across the interface to ensure the continuity in amplitude and pressure there.

We take the particular model for which

$$b = .5$$

$$\varrho/\sigma = 10$$

as the ratio of the time period of the fundamental to that of the 1st overtone in thi case is 2.18

The values of σ_k's for the 1st three modes as given by C. Prasad (1953) are

$$\sigma_1 = \frac{1}{\cdot 5024} 2\pi(G\bar{\varrho}/6\pi)^{\frac{1}{2}}$$

$$\sigma_2 = \frac{1}{\cdot 2302} 2\pi(G\varrho/6\pi)^{\frac{1}{2}}$$

$$\sigma_3 = \frac{1}{\cdot 1375} 2\pi(G\varrho/6\pi)^{\frac{1}{2}}$$

$\bar{\varrho}$ here refers to the mean density of the star.

With the help of these values of σ_k's we evaluate η_k's and η'_k's from the solutions o equations (12) and (13) for the first three modes correct to four places of the decimal.

We consider the 1st three modes for which equations (5) can be written

$$\left.\begin{aligned}
\frac{d^2q_1}{d\tau^2} + q_1 &= \frac{\gamma}{\sigma_1^2 I_1}[D_{111}q_1^2 + 2D_{121}q_1q_2 + 2D_{131}q_1q_3],\\
\frac{d^2q_2}{d\tau^2} + \beta_2 q_2 &= \frac{\gamma}{\sigma_1^2 I_1}[D_{112}q_1^2 + 2D_{122}q_1q_2 + 2D_{132}q_1q_3],\\
\frac{d^2q_3}{d\tau^2} + \beta_3 q_3 &= \frac{\gamma}{\sigma_1^2 I_3}[D_{113}q_1^2 + 2D_{123}q_1q_2 + 2D_{133}q_1q_3],
\end{aligned}\right\} \quad (14)$$

The terms containing q_2^2, q_3^2 and $q_2 q_3$ have been neglected as their contribution will be found to be very small. On evaluating the coefficients D_{ijk} and I_k for $\gamma = \frac{5}{3}$ from the columns of η_k and η'_k by means of numerical integrations equations (14) reduce to

$$\left.\begin{aligned}
\ddot{q}_1 + q_1 &= 2{\cdot}57867q_1^2 - 4{\cdot}02803q_1q_2 - {\cdot}00096q_1q_3\\
\ddot{q}_2 + 4{\cdot}7630qq_2 &= -1{\cdot}460422q_1^2 + 26{\cdot}90452q_1q_2 + 6{\cdot}13406q_1q_3\\
\ddot{q}_3 + 13{\cdot}35038 &= -{\cdot}0014466q_1^2 + 25{\cdot}37914q_1q_2 + 126{\cdot}53000q_1q_3
\end{aligned}\right\} \quad (15)$$

Substituting (10) and (15) we obtain the coefficients $a_0, a_2, a_3, \ldots; b_0, b_1, b_2, \ldots;$ $c_0, c_1, c_2, \ldots$ in term of a_1 which is chosen to be ·05, by the iteration process which i repeated five times to ensure the convergence of the coefficients. That gives the solutions of (15) as

$$\left.\begin{aligned} q_1 &= 0{\cdot}0038530 + 0{\cdot}05 \cos n\tau - 0{\cdot}0011253 \cos n\tau - 0{\cdot}0000176 \cos 3n\tau \\ q_2 &= -0{\cdot}0005198 - 0{\cdot}0008462 \cos n\tau - 0{\cdot}002011 \cos 2n\tau + 0{\cdot}0004662 \cos 3n\tau \\ q_3 &= -0{\cdot}0001290 - 0{\cdot}0003806 \cos n\tau - 0{\cdot}0004335 \cos 2n\tau - 0{\cdot}0004406 \cos 3n\tau \end{aligned}\right\} \qquad (16)$$

where $$n^2 = 0{\cdot}97705$$

The total displacement at the surface of the star will be

$$\eta(R) = \Sigma q_k = 0{\cdot}002704 + 0{\cdot}048773 \cos n\tau - 0{\cdot}0044707 \cos 2n\tau - 0{\cdot}0002920 \cos 3n\tau$$

For mean amplitude ·05 the skewness in the velocity time curve is found to be 1·426 and the increase in the period to be 1·154% of the period for the fundamental mode for vanishing amplitude.

The following graphs represent amplitude and velocity at the surface as functions of time.

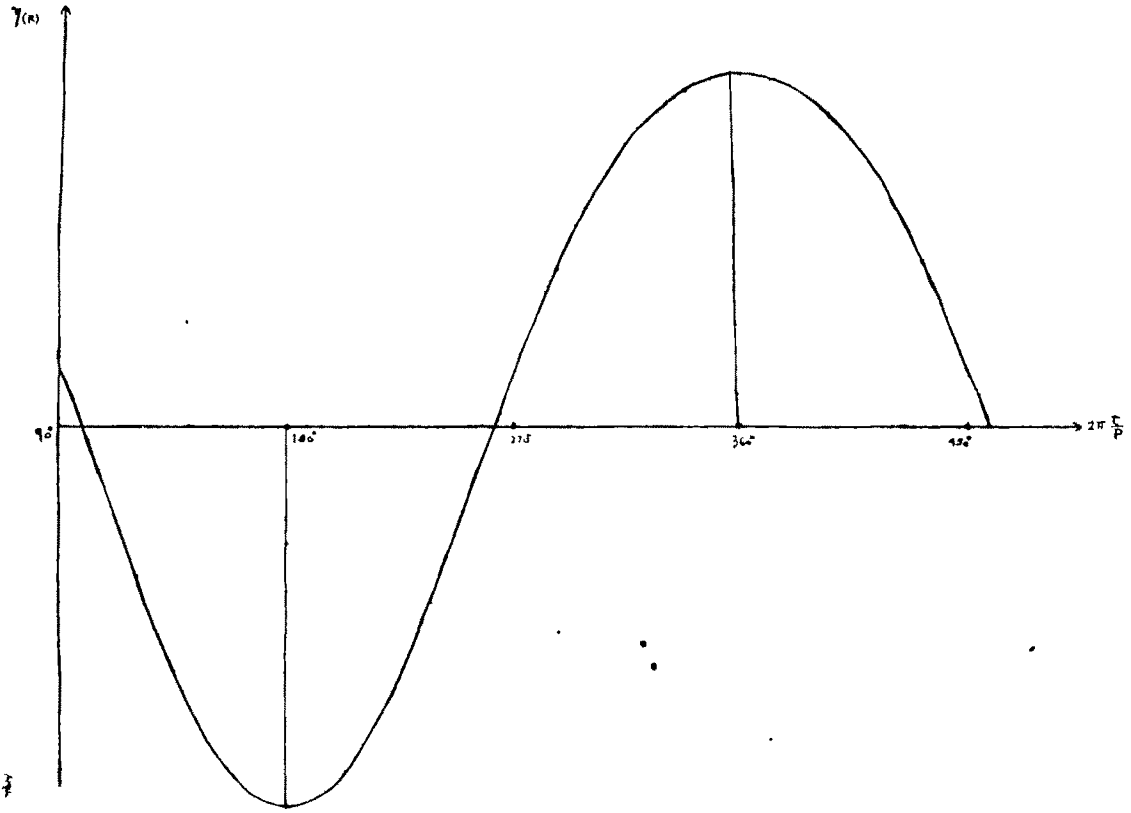

Fig. 1. The graph represents the surface amplitude as function of time. The abcissa represents $2\pi t/P$, where P is the period of pulsation and ordinates represents surface amplitude $\eta_{(R)}$.

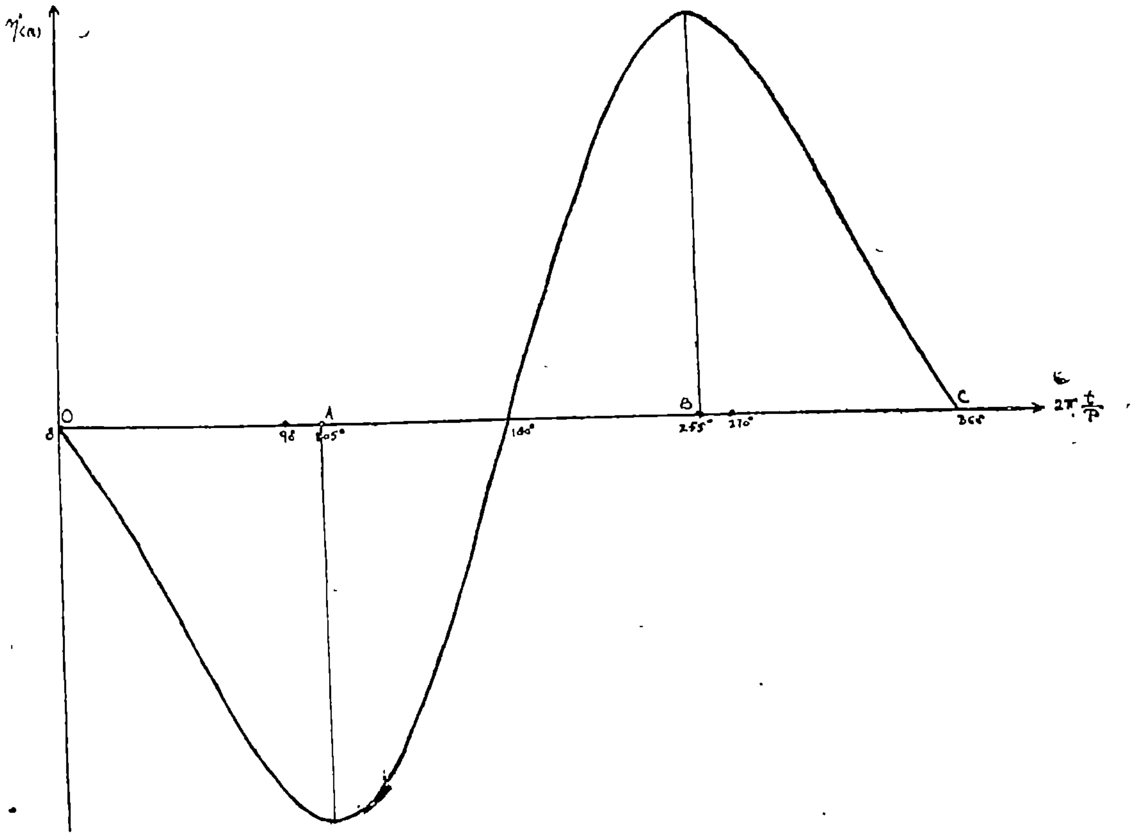

Fig. 2. The graph represents the velocity at the surface as function of time. The skewness $AB/(OA+BC)$ of the velocity-time curve is 1.426.

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